Hw 7 Solutions

Note Title 4/14/2009

11.3(4)

Let $L = \lim_{x \to x_0} f(x)$

Take $\varepsilon = M - L$

Then, by def of limit:

$$f(x) \underset{M-L}{\approx} L \quad \text{for} \quad x \underset{\neq}{\approx} x_0$$

$$\Rightarrow \quad f(x) < L + (M - L) = M$$

$$\text{for} \quad x \underset{\neq}{\approx} x_0$$

$\square$

11.4(2)

Want to show:

$$\underset{\mathrm{I}}{\begin{pmatrix} a < x_0 < b \\ \Downarrow \\ f(a) < f(x_0) \end{pmatrix}} \quad \underset{\mathrm{III}}{\text{and}} \quad \underset{\mathrm{II}}{\begin{pmatrix} a < x_0 < b \\ \Downarrow \\ f(x_0) < f(b) \end{pmatrix}}$$

$$\text{and } \big(f(a) < f(b)\big)$$

pf of I

Suppose $a < x_0 < b$

Pick $a < x_1 < x_0$

We know $f(x_1) < f(x_0)$

and for $a < x < x_1$

$$f(x) < f(x_1)$$

By limit location thm

$$f(a) = \lim_{x \to a^+} f(x) \le f(x_1) < f(x)$$

limit form of continuity

pf of part II

Suppose $a < x_0 < b$

Choose $x_0 < x_1 < b \Rightarrow f(x_0) < f(x_1)$

For $x_1 < x < b \Longrightarrow f(x_1) < f(x)$

$\Longrightarrow$ (limit location) $\quad f(b) = \lim_{x \to b^-} f(x) \ge f(x_1) > f(x_0)$

limit form of continuity

pf of part III

Pick $x_0 \in (a,b)$

$$\underset{\text{part I}}{f(a) < f(x_0)} \underset{\text{part II}}{<} f(b)$$

11.5(1)

(a)

Let x be irrational.

Choose a sequence of rational numbers $\{x_n\}$ such that $x_n \to x$.

Such a sequence exists: for instance, if x has decimal expansion

$$x = a_1 a_2 a_3 \cdots a_k . b_1 b_2 b_3 \cdots\cdots$$

then one could take

$$x_n = a_1 a_2 \cdots\cdots a_k . b_1 b_2 b_3 \cdots b_n$$

(finite decimal)

By the sequential continuity thm

$$\underset{\overset{\shortparallel}{0}}{f(x_n)} \longrightarrow f(x)$$

$$\Longrightarrow f(x) = 0$$

(b) Again, let x be irrational,
and suppose $x_n \to x$
with $x_n \in \mathbb{Q}$ for all n.

$$f(x_n) \leq g(x_n)$$

and $f(x_n) \longrightarrow f(x)$ [Sequential cont. Thm]
$g(x_n) \longrightarrow g(x)$

$\Longrightarrow$ $f(x) \leq g(x)$
(limit location thm)

However, consider

$$f(x) = 0$$
$$g(x) = (x-\pi)^2$$

$g(x) > f(x)$ for all rational x
(in fact all $x \neq \pi$)

but $g(\pi) \not> f(\pi)$.

12.1 (1)

Suppose $f(x) \in \mathbb{Q}$ for all x,

f is continuous,

but f is not constant.

Then there exist $a < b$

such that $f(a) \neq f(b)$.

There exists an irrational number y

between $f(a)$ and $f(b)$.

IVT $\Rightarrow$ there exists $x \in [a, b]$ such

that $f(x) = y \notin \mathbb{Q}$

$\rightarrow\leftarrow$

So f must be constant.

12.1(5)   If $y = 1$:

$y^2 \cos(0) - e^0 = 0$, and we're done.

Assume $y > 1$.

$\Rightarrow y^2 \cos(0) - e^0 > 0$

Also $y^2 \cos(\pi/2) - e^{\pi/2} = -e^{\pi/2} < 0$

$f(x) = y^2 \cos(x) - e^x$ is continuous in x,

IVT $\Rightarrow$ there is $x_0 \in [0, \pi/2]$ such that

$f(x_0) = y^2 \cos(x_0) - e^{x_0} = 0$

Since $f(\pi/2) < 0$, $x_0 \neq \pi/2$.

12-2:

Since f never repeats a value,

$f(a) < f(b)$

Let $a \le x' < x'' \le b$

Suppose $f(x') \ge f(x'')$

By assumption, since $f(x') \ne f(x'')$

$$f(a) < f(x'') < f(x')$$

IVT $\Rightarrow$ there exists $x_0 \in [a, x']$

such that $f(x_0) = f(x'')$

but since $x'' > x'$, $x_0 \ne x''$.

This contradicts the fact that

f never repeats a value.

Thus, $f(x') < f(x'')$.