Hw 8 Solutions

Note Title 4/14/2009

13.1(1)

(a) Suppose $S = I_1 \cup I_2 \cup \cdots \cup I_k$

I_j a compact interval.

Let $\{x_n\}$ be a sequence in S.

Then must exist at least one

$1 \le p \le k$

such that I_p contains infinitely many x_n's.

Let $\{x_{n_i}\}$ consist of the subsequence of terms of $\{x_n\}$ that lie in I_p.

By the sequential compactness thm, there exists a subsequence

$\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$

such that

$$x_{n_{i_j}} \longrightarrow L \in I_p$$

But $\{x_{n_{i_j}}\}$ is also a subsequence of $\{x_n\}$, and $L \in S$.

Thus we have shown S is sequentially compact.

(b) Consider $S = [0,1] \cup [1,2] \cup [2,3] \cup \cdots$

$S = [0, \infty)$

$\{x_n\} = \{1, 2, 3, \ldots\}$

is an example of a sequence in S with No convergent subsequences.

$\Longrightarrow$ S not sequentially compact.

13.2 (1)

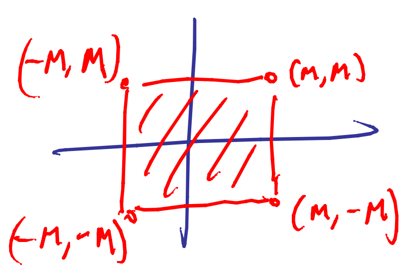
By boundedness thm

$x(t)$, $y(t)$ are bounded on $[a, b]$

Therefore, there is $M > 0$ such that

$$-M \le x(t) \le M, \quad -M \le y(t) \le M \quad \text{for } t \in [a, b]$$

$\Longrightarrow$ $(x(t), y(t))$ lies in



13.3(1)

(a) Suppose $m = \min_{\mathbb{R}} f(x)$

Since $f(x) > 0$, we deduce $m > 0$.

However, since $\lim_{x \to \infty} f(x) = 0$

$\Rightarrow$ for $\varepsilon = m$, there is M

such that

$|f(x)| < m$. This contradicts the fact that m is a minimum.

$\overset{\shortparallel}{f(x)}$

Thus f does not have a minimum.

(b) Since $\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

there is an $M > 0$

such that $(\varepsilon = f(0))$;

$$f(x) < f(0) \text{ for } x > M$$

$$f(x) < f(0) \text{ for } x < -M$$

By the maximum theorem, there is an $x_0 \in [-M, M]$ such that

$$f(x_0) = \max_{[-M,M]} f(x)$$

but if $x \notin [-M, M]$

$$f(x) < f(0) \leq \max_{[-M,M]} f(x) = f(x_0)$$

Thus $f(x_0) = \max_{(-\infty,\infty)} f(x).$

13.5(i)

Recall that $|\sin(x') - \sin(x'')| < |x' - x''|$

Take $\varepsilon > 0$.

for $x' \approx_\varepsilon x''$,

$|\sin(x') - \sin(x'')| < |x' - x''| < \varepsilon$

$\Rightarrow \sin(x') \approx_\varepsilon \sin(x'')$.

13-4 Let $\overline{M} = \operatorname{Max} f$

$\underline{M} = \operatorname{Min} f$

Let $\{x_n\}$ be an infinite sequence in I such that $x_n \neq x_m$ for all $n \neq m$, and

$$f(x_n) = \overline{M}$$

By sequential compactness thm, there exists a subsequence $\{x_{n_i}\}$ so that $x_{n_i} \longrightarrow L \in I$

By assumption, for every i

there exists y_i between x_{n_i} and $x_{n_{i+1}}$

such that $f(y_0) = \underline{M}$.

Claim: $y_i \longrightarrow L$.

Indeed, take $\varepsilon > 0$.

Since $x_{n_i} \longrightarrow L$, there is an $N > 0$

such that for all $i > N$

$$x_{n_i} \approx_\varepsilon L$$

But then, for $i > N$,

x_{n_i} and $x_{n_{i+1}}$ are in

$$(L-\varepsilon, L+\varepsilon).$$

$$\Longrightarrow y_i \in (L-\varepsilon, L+\varepsilon)$$

$$\Longrightarrow y_i \approx_\varepsilon L \quad \text{for } i \gg 0$$

$\Rightarrow \quad y_i \longrightarrow L.$

Now, by sequential continuity theorem,

$$\underset{\underline{M}}{\overset{\shortparallel}{f(y_i)}} \longrightarrow f(L)$$

$$\underset{\overline{M}}{\overset{\shortparallel}{f(x_{n_i})}} \longrightarrow f(L)$$

$\Rightarrow \quad \overline{M} = f(L) = \underline{M}$

$\Rightarrow \quad \min f = \max f$

$\Rightarrow \quad f$ constant.

$\sin(1/x)$ is not a counterexample because in order for it to have infinitely many maxima and minima, I would have to contain 0.

But $\sin(1/x)$ cannot be extended continuously to $x = 0$.