## 18.100A Practice Exam

This exam is long. Try to do as many problems as you can, the ones you feel most comfortable with first. They are not in the order of difficulty. A problem or a part of a multiple problem counts 10 points.

You can use the book but no notes, problem sets, calculators, etc.

Cite theorems used, by name or number.

1. Let $S_1, S_2, \ldots, S_n$ be a finite set of compact intervals on the line. Prove their union $S = S_1 \cup S_2 \cup \cdots \cup S_n$ is a sequentially compact subset of the line.

(Use methods of early chapter; no credit for using or quoting any theorems from Chapter 25).

2. Prove using the properties of integral given in Chapter 19: If $f(x)$ is continuous and non-negative on an interval $[a,b]$ and $f(c) > 0$ for some point in $[a,b]$, then $\int_a^b f(x)dx > 0$.

3. Assume that functions $g_n(x)$, $n = 0, 1, 2, \ldots$, are all continuous on $[a,b]$ and that on this interval they converge uniformly to $g(x)$. Prove that on this interval $[a,b]$, their indefinite integrals also converge uniformly:

$$\int_a^x g_n(t)dt \rightrightarrows \int_a^x g(t)dt.$$

4. Prove: $f(x) = \begin{cases} 0, & x \text{ rational} \\ 1, & x \text{ irrational} \end{cases}$ is not continuous at any point $a$.

(Hint: assume it is continuous at some point $a$ and use sequential continuity to get a contradiction.)

5. Prove $f(x) = \sum_1^\infty \frac{x}{n(x+n)}$ is continuous on $[0, \infty)$.

6. (a) Prove: given $\epsilon > 0$, $\ln x \quad (x^\epsilon - 1)/\epsilon$ for $x > 1$.

(b) Deduce from (a) that $\lim_{x\to\infty} \frac{\ln x}{x^\epsilon} = 0$ for any $\epsilon > 0$.

7. [redacted] circle. (Use sequential continuity.)

8. Let $S$ be a [redacted] set of points in the plane which is [redacted]

(Here "closest" means that no other point of $S$ is closer to the origin.)

9. Prove the case $L > 1$ of the ratio test for series convergence.

[redacted]

11. Does $\displaystyle\int_0^\infty \frac{1}{\sqrt{t(1+t^2)}}dt$ converge or diverge? Justify your answer.

12. Let $f(x) = \displaystyle\sum_1^\infty \frac{\sin nx}{(n-1)!}$. Prove that $\displaystyle\int_0^\pi f(x)dx$ exists and evaluate it.