18.100A Practice Midterm I – Answers.

1. Given $\epsilon > 0$, $\left|\dfrac{3n^2+1}{n^2+1} - 3\right| = \dfrac{2}{n^2+1}$ and $\dfrac{2}{n^2+1} < \epsilon$ if $n^2+1 > 2/\epsilon$ or $n > \sqrt{2/\epsilon - 1}$.

2. The first terms of the sequence are $0, 3/2, -2/3, 5/4, -4/5$.

max = sup = 3/2, min does not exist, inf = −1. Cluster points: −1, 1. Subsequences: $a_{2n} = 1 + \frac{1}{2n} \to 1$, $a_{2n+1} = -1 + \frac{1}{2n+1} \to -1$.

3. Using ratio test: $\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\dfrac{x^{n+1}}{2^{3n+1}\sqrt{n+1}} \dfrac{2^{3n}\sqrt{n}}{x^n}\right| = \dfrac{|x|}{8}\sqrt{\dfrac{n}{n+1}} \to \dfrac{|x|}{8} < 1 \Leftrightarrow |x| < 8$. So, $R = 8$.

At $x = R$: $\displaystyle\sum_1^\infty \frac{8^n}{2^{3n}\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ diverges.

At $x = -R$: $\displaystyle\sum_1^\infty \frac{(-8)^n}{2^{3n}\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$ converges by Cauchy's test: terms alternate in sign, $1/\sqrt{n}$ is decreasing, $1/\sqrt{n} \to 0$.

4. $\dfrac{a_{n+1}}{a_n} = \dfrac{c^{n+1}}{(n+1)!}\dfrac{n!}{c_n} = \dfrac{c}{n+1}$, ≤ 1 if $n > c-1$. Thus $a_{n+1} < a_n$ if $n > c - 1$. Decreasing for $n \gg 1$.

5. Assume $L \neq M$. Choose $\epsilon = |L - M| > 0$. Then $|L - M| < \epsilon$ is false. Contradiction. Thus $L = M$.

6. $m = \max a_n$. Claim: $a_n = m$ for $n \gg 1$.

(if only claim $\lim a_n = m$, get less credit)

Proof: By hypothesis $a_N = m$ for some m. Then if $n > N$, $a_n \leq m$ because m is maximum. AND $a_n \geq a_N = m$ because $\{a_n\}$ increasing. Thus $a_n = a_N$ if $n > N$.

7. $a \leq x_n \leq b$ for all n. Bounded. By Bolzano–Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $x_{n_i} \to c$.

$x_{n_i} \leq b$ for all $i \Rightarrow c \leq b$;

$x_{n_i} \geq a$ for all $i \Rightarrow c \geq a$.

(Both statements by Limit Location Theorem.)

Thus $c \in [a, b]$.

8. a) $\sum a_n$ absolutely convergent $\Rightarrow \sum |a_n|$ convergent $\Rightarrow |a_n| \to 0$ (n-th term theorem) $\Rightarrow |a_n| < 1$ for $n \gg 1$ (Sequence location theorem) $\Rightarrow a_n^2 = |a_n^2| \leq |a_n|$ for $n \gg 1$ (say $n \geq N$) $\Rightarrow \sum_N^\infty a_n^2 \leq \sum_N^\infty |a_n| \Rightarrow \sum_N^\infty a_n^2$ converges (Comparison theorem) $\Rightarrow \sum a_n^2$ converges (Tail-convergence theorem).

b) $a_n = (-1)^/\sqrt{n}$. $\sum a_n$ converges (see sol-n for problem 3). $\sum a_n^2 = \sum 1/n$ diverges (harmonic series).

9. Need to verify that 0 satisfies **inf-1** and **inf-2** for $T = \{\bar{m} - x : x \in S\}$.

inf-1: $x \leq \bar{m}$ for all $x \in S$. Thus $0 \leq \bar{m} - x$. Conclusion: 0 is a lower bound for T.

inf-2: b is a lower bound for T. $b \leq \bar{m} - x$ for all $\bar{m} - x \in T \Rightarrow x \leq \bar{m} - b$ for all $x \in S \Rightarrow \bar{m} - b$ is an upper bound of $S \Rightarrow \bar{m} = \sup S \leq \bar{m} - b \Rightarrow b \leq 0$.