HOMOLOGY AND COHOMOLOGY

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1. Introduction

We have been introduced to the idea of homology, which derives from a chain complex of singular or simplicial chain groups together with some map ∂ between chain groups $C_n \to C_{n-1}$. The map ∂ has the property that $\partial\partial\phi = 0$ for all chains ϕ. We define the nth homology group $H_n(X)$ for this chain complex to be the quotient group $\ker\partial/\text{Im }\partial$ at C_n.

We now shift our concern from these chain groups C_n to dual cochain groups C_n^*. We define each dual cochain group C_n^* to be the respective group of all homomorphisms $\text{Hom}(C_n, G)$ between C_n and a selected group G. We can define a dual coboundary map $\delta : C_n^* \to C_{n+1}^*$ sending a homomorphism $\varphi : C_n \to G$ to the homomorphism $\varphi\partial : C_{n+1} \to G$. As $\partial\partial = 0$, $\delta\delta$=0, and the cohomology group $H^n(C; G)$ is the quotient group $\ker\delta/\text{Im }\delta$ at C_n^*.

2. Motivation

The motivation here comes from observing the set of all functions f from basis elements of C_n to some selected group G. Because these functions are defined on basis elements, this set of functions is identified precisely with the set of all homomorphisms of C_n to G.

Consider that we are observing a one-dimensional Δ-complex X, where $\Delta_1(X)$ and $\Delta_0(X)$ are the simplicial chain groups. We can have a function f that defines the height of each vertex of the complex as an integer; this function extends naturally to a homomorphism $f : \Delta_0(X) \to \mathbb{Z}$. The coboundary of f, δf may then be the map that calculates the change in height along each oriented edge of the complex. $\delta f([v_0, v_1]) = f([v_1]) - f([v_0])$. This extends naturally to a homomorphism $\delta f : \Delta_1(X) \to \mathbb{Z}$.

The kernel of the coboundary homomorphism of the group of 1-cochains is the entire group, since the group of 2-cochains is empty. The cohomology group $H^1_\Delta(X; \mathbb{Z})$ is then the quotient of the entire group of 2-cochains with the boundary (image) of the group of 1-cochains. The nontrivial elements are generated by the functions from edges that are not the images of functions from vertices.

If X is a tree, or a disjoint union of trees, then for any function g from the edges to $\mathbb{Z}$ we can find a function f whose coboundary is g by defining the value of f for one vertex of each connected component of X and then extrapolating values for all neighboring vertices using the difference values that g defines along each edge. Induction gives the values of f for all vertices in X. Every basis element of the group of 1-cochains has a pre-image in the group of 0-cochains, so the coboundary homomorphism is surjective and the cohomology group $H^1_\Delta(X; Z)$ is trivial.

Otherwise, we can define f such that $\delta f = g$ if and only if the following condition holds. For any loop of edges $\{[v_0, v_1], ..., [v_{n-1}, v_n], [v_n, v_0]\} \subset \Delta_1(X)$,

$$0 = g([v_0, v_1]) + ... + g([v_{n-1}, v_n]) + g([v_n, v_0]).$$

Those g that have nontrivial cohomology classes are those that do not satisfy this condition.

The element $([v_0, v_1] + ... + [v_{n-1}, v_n] + [v_n, v_0]) \in \Delta_1(X)$ is in fact an elementary "cycle" in $\Delta_1(X)$, that is, it is in $\ker \partial$. It is always true that all "coboundaries" δf must vanish on cycles. In this particular instance the converse is true: all homomorphisms that vanish on cycles are coboundaries. This second statement is not always true for higher cochain groups.

3. Background

Before discussing cohomology, we first recall some familiar definitions from homology.

Definition 3.1. A **chain complex** is a sequence

$$... \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \tag{3.1.1}$$

of chain groups together with **boundary homomorphisms** ∂_i between the chain groups with the important property that $\partial_i \partial_{i+1} = 0$ for all i. For ease of notation, we frequently refer to each boundary homomorphism simply as ∂, without the subscript.

With this property, we know that $\text{Im } \partial_{n+1} \subset \ker \partial_n$. We call elements of the image **boundaries**, and elements of the kernel **cycles**. The **nth homology group** in the chain complex is defined to be the quotient

$$H_n(C) = \ker \partial_n / \text{Im } \partial_{n+1}. \tag{3.1.2}$$

Definition 3.2. A sequence

$$... \xrightarrow{g_{n+2}} G_{n+1} \xrightarrow{g_{n+1}} G_n \xrightarrow{g_n} G_{n-1} \xrightarrow{g_{n-1}} ... \tag{3.2.1}$$

of groups together with homomorphisms g_i between them is said to be **exact at $\mathbf{G_n}$** if $\text{Im } g_{n+1} = \ker g_n$. A sequence that is exact at every stage is said to be an **exact sequence**.

A **short exact sequence** is an exact sequence with 0's at either end:

$$0 \to G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \to 0. \tag{3.2.2}$$

A **split short exact sequence** is a short exact sequence where $G_1 \cong G_2 \oplus G_0$.

Note that for a short exact sequence, the homomorphism g_2 must be injective, and the homomorphism g_1 must be surjective.

4. Short exact sequence of chain complexes

One concept that we will use repeatedly when discussing cohomology will be the idea that any short exact sequence of chain complexes can be stretched out into a long exact sequence of homology groups. While we will explain this concept for a short exact sequence of chain complexes and homology groups, it also extends naturally to a short exact sequence of cochain complexes and cohomology groups.

A commutative diagram of the form

$$(4.0.3)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \xrightarrow{\partial} & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{\partial} & \ldots \\ & & \downarrow i & & \downarrow i & & \downarrow i & & \\ \ldots & \xrightarrow{\partial} & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \xrightarrow{\partial} & \ldots \\ & & \downarrow j & & \downarrow j & & \downarrow j & & \\ \ldots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where the columns are exact and the rows are chain complexes is said to be a **short exact sequence of chain complexes**.

We will show that we can stretch out any short exact sequence of chain complexes into a **long exact sequence of homology groups**:

$$(4.0.4)\qquad \ldots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} \ldots$$

Note that these groups $H_n(G)$ are the homology groups of the homomorphisms $\partial : G_n \to G_{n-1}$, where "$G$" can be replaced by "$A$", "$B$", or "$C$".

The commutativity of the squares in the short exact sequence of chain complexes implies that i and j are chain maps: $i(\partial a) = \partial i(a)$ and $j(\partial b) = \partial j(b)$. The image of any boundary is a boundary, and the image of any cycle is a cycle. They induce homomorphisms $i_* : H_n(A) \to H_n(B)$ and $j_* : H_n(B) \to H_n(C)$. We now must define $\partial : H_n(C) \to H_{n-1}(A)$.

Since j is onto, $c = j(b)$ for some $b \in B_n$. $\partial b \in B_{n-1}$ is in $\ker j$, as can be seen by a direct calculation $j(\partial b) = \partial j(b) = \partial c = 0$. Since $\ker j =$Im i, $\partial b = i(a)$ for some $a \in A_{n-1}$.

Note: $i(\partial a) = \partial i(a) = \partial\partial b = 0$. by injectivity of i, $\partial a = 0$. Thus we know a is a cycle, and hence has a homology class $[a]$.

Lemma 4.1. *Define $\partial : H_n(C) \to H_{n-1}(A)$ to be the map sending the homology class of c to the homology class of a as defined above. $\partial[c] = [a]$. ∂ is a well defined homomorphism between homology groups.*

Proof. ∂ is well-defined:

* a is uniquely determined by ∂b since i is injective.

* if we choose a different value b' instead of b where $j(b') = j(b) = c$, then $j(b' - b) = 0$ so $b' - b \in \ker j =$Im i. $b' - b = i(a')$ for some a', so we can write $b' = b + i(a')$. Our value a is then replaced by $a + \partial a'$: $i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial i(a') = \partial(b + i(a')) = \partial b'$. This new element $a + \partial a'$ is homologous to a so we get the same homology class for $\partial[c]$.

* if we choose a different value c' in the homology class of c, we replace c with $c' = c + \partial c''$ for some value $c'' \in C_{n+1}$. $c'' = j(b'')$ for some $b'' \in B_{n+1}$. $c' = c + \partial j(b'') = c + j(\partial b'') = j(b + \partial b'')$.

b is replaced by $b + \partial b''$, which leaves ∂b unchanged and therefore we get the same value for a.

$\partial : H_n(C) \to H_{n-1}(A)$ is a homomorphism:

Let $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$ via elements b_1 and b_2 as above. $j(b_1+b_2) = j(b_1)+j(b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. Thus calculating $\partial([c_1] + [c_2])$ as above gives $[a_1] + [a_2]$, as required. □

Theorem 4.2. *The long sequence of homology groups*

$$... \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} ...$$

is exact.

Proof. We must verify that the kernel of each homomorphism is equal to the image of the prior homomorphism.

Im $\mathbf{i_*}$ = ker $\mathbf{j_*}$: Im $i_* \subset \ker j*$ because $ji = 0$ in the short exact sequence and this implies that $j_*i_* = 0$.

Consider now a representative cycle $b \in B_n$ for a homology class in the kernel of j_*. $j_*([b]) = 0$, so $j(b) = \partial c'$ for some $c' \in C_{n+1}$. Since j is surjective at each dimension, $c' = j(b')$ for some $b' \in B_{n+1}$. We evaluate $j(b - \partial b') = j(b) - j(\partial b') = j(b) - \partial j(b') = j(b) - \partial c' = j(b) - j(b) = 0$. $(b - \partial b') \in \ker j =$Im i. Thus $b - \partial b' = i(a)$ for some $a \in A_n$. $i(\partial a) = \partial i(a) = \partial(b - \partial(b')) = \partial b = 0$ since b is a cycle. By injectivity of i, then, $\partial a = 0$ and a is a cycle with a homology class $[a]$. $i_*([a]) = [b - \partial b'] = [b]$, so $[b] \in$Im i_*.

Im $\mathbf{j_*}$ = ker ∂: If $[c] \in$Im j_*, then b as defined when calculating $\partial[c]$ has a homology class and is therefore a cycle. $\partial b = 0$ so $\partial([c]) = [a] = 0$. Thus $[c] \in \ker \partial$.

Now assume $[c]$ is in $\ker \partial$. $\partial[c] = [a] = 0$ so $a = \partial a'$ for some $a' \in A_n$. $\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = \partial b - \partial b = 0$, so the element $(b - i(a'))$ is a cycle in B_n and has a homology class $[b - i(a')]$. $j(b - i(a')) = j(b) - ji(a') = j(b) = c$ so $j_*([b - i(a')]) = [c]$ and thus $[c] \in$Im j_*.

Im ∂ = ker $\mathbf{i_*}$: i_* takes $\partial[c] = [a]$ to $[\partial b]$, which is 0, so Im $\partial \subset \ker i_*$.

A homology class in $\ker i_*$ is represented by an cycle $a \in A_{n-1}$ where $i(a) = \partial b$ for some $b \in B_n$. $\partial j(b) = j(\partial b) = ji(a) = 0$, so $j(b)$ is a cycle and has a homology class $[j(b)]$. The homomorphism ∂ takes $[j(b)]$ to $[a]$, and thus $[a] \in$Im ∂. □

We now have established a good deal of machinery and can begin our discussion of cohomology groups.

5. Cohomology and the Universal Coefficient Theorem

Definition 5.1. Cohomology.

Given a chain complex

$$... \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ...$$

and a group G, we can define the **cochains** C_n^* to be the respective groups of all homomorphisms from C_n to G:

$$C_n^* = Hom(C_n, G). \tag{5.1.1}$$

We define the **coboundary map** $\delta_n : C_{n-1}^* \to C_n^*$ dual to ∂_n as the map sending $\varphi \mapsto \delta_n\varphi = \partial_n^*\varphi$. For an element $c \in C_n$ and a homomorphism $\varphi \in C_{n-1}^*$, we have

$$\delta_n\varphi(c) = \varphi(\partial_n c). \tag{5.1.2}$$

Because $\partial_n\partial_{n+1} = 0$, it is easily seen that $\delta_{n+1}\delta_n = 0$. In other words, Im $\delta_n \subset \ker\delta_{n+1}$. With this fact we can define the **nth cohomology group** as the quotient:

$$H^n(C; G) = \ker\delta_{n+1}/\text{Im }\delta_n. \tag{5.1.3}$$

Again, for ease of notation, we will simply use the symbol δ to denote a coboundary map and will not usually use the subscripts.

Define Z_n to be the group of cycles in C_n ($\ker\partial_n$), and B_n to be the group of boundaries in C_n (Im ∂_{n+1}). As discussed earlier, $B_n \subset Z_n$.

A cohomology class in $H^n(C; G)$ is represented by a homomorphism $\varphi : C_n \to G$ such that $\delta\varphi = 0$. $\varphi\partial = 0$, or φ vanishes on B_n. Cocycles in C_n^* are precisely those homomorphisms that vanish on all boundaries in C_n.

If φ is a coboundary $\varphi = \delta\psi$ for some $\psi \in C_{n-1}^*$, then $\varphi(c) = \delta\psi(c) = \psi(\partial(c))$, so φ vanishes on Z_n. Coboundaries in C_n^* all vanish on cycles in C_n, but the converse is frequently not true.

Consider the quotient of the subgroup of homomorphisms in C_n^* vanishing on B_n with the subgroup of homomorphisms in C_n^* vanishing on Z_n (note that we can always define quotient group because a homomorphism vanishing on all cycles must vanish on the subgroup of boundaries); temporarily call this quotient group $J^n(C; G)$. Homomorphisms in this quotient group are identified with homomorphisms in $Hom(Z_n/B_n, G) = Hom(H_n(C), G)$.

Assume that for a given chain complex C and group G, every homorphism $\varphi : C_n \to G$ that vanishes on a cycle in C_n is a coboundary. In this case the coboundaries are precisely those functions that vanish on Z_n, and we already know that the cocycles are precisely those functions that vanish on B_n. Hence $H^n(C; G) = J^n(C; G)$ and in this case it is isomorphic with $Hom(H_n(C), G)$.

5.2. **Motivation for the Universal Coefficient Theorem.** An extremely important tool in cohomology is the Universal Coefficient Theorem, which completely determines the nth cohomology group $H^n(C; G)$ of a chain complex C and group G, from the nth homology group $H_n(C)$ and the group G.

As above, if the homomorphisms vanishing on cycles are all coboundaries, then the groups $Hom(H_n(C), G)$ and $H^n(C; G)$ are isomorphic. The statement of the Universal Coefficient Theorem provides a measure of the failure of this statement to be true.

Definition 5.3. The map h.

It would be ideal if $H^n(C; G)$ were isomorphic to $Hom(H_n(C), G)$. Unfortunately, this is not usually the case. There is, however, a natural map $h : H^n(C; G) \to Hom(H_n(C), G)$.

Given a representative cocycle φ for a cohomology class in $H^n(C; G)$, consider the restriction $\varphi_0 = \varphi \mid Z_n$. This induces a quotient homomorphism $\bar{\varphi_0} : Z_n/B_n \to G$. $\bar{\varphi_0} \in Hom(H_n(C), G)$. If the cohomology class of φ is [0], then φ is a coboundary, and as discussed earlier, φ vanishes on Z_n, so the restriction homomorphism $\varphi_0 = 0$ and the induced quotient homomorphism $\bar{\varphi_0} = 0$. Thus there is a well-defined map:

$$h : H^n(C, G) \to Hom(H_n(C), G) \tag{5.3.1}$$

sending

$$[\varphi] \mapsto \bar{\varphi_0}. \tag{5.3.2}$$

The map h is a homomorphism.

Lemma 5.4. *The homomorphism h is surjective.*

Proof.

The exact sequence

$$0 \to Z_n \to C_n \xrightarrow{\partial} B_{n-1} \to 0 \tag{5.4.1}$$

splits, since B_{n-1} is free, being a subroup of the free abelian group C_{n-1}. There is thus a projection homomorphism $p : C_n \to Z_n$ that restricts to the identity on Z_n.

Given a homomorphism $\varphi_0 : Z_n \to G$, we see that $\varphi_0 p$ is a homomorphism from C_n to G. $\varphi_0 p(c) = \varphi_0(c)$ for all $c \in Z_n$.

There is a one-to-one correspondence of homomorphisms from $H_n(C)$ to G (elements of $Hom(H_n(C), G)$) and homomorphisms $\varphi_0 : Z_n \to G$ that vanish on B_n. If $\varphi_0 : Z_n \to G$ vanishes on B_n, $\varphi_0 p$ vanishes on B_n as well and it is thus a cocycle, $\varphi_0 p \in \ker \delta$. Thus we have a natural homomorphism from the group $Hom(H_n(C), G)$ to the group $\ker \delta$.

If we compose this homomorphism with the quotient map $\ker \delta \to H^n(C; G)$, we have a homomorphism from $Hom(H_n(C), G)$ to $H^n(C; G)$. Following this homomorphism with h gives the identity on $Hom(H_n(C), G)$:

We first send a homomorphism $\varphi_{-1} : H_n(C) \to G$ to the single homomorphism $\varphi_0 : Z_n \to G$ that vanishes on B_n and takes the values for other elements of Z_n that φ_{-1} takes for their quotient classes in $H_n(C)$. We then extend φ_0 to a homomorphism $\varphi_1 : C_n \to G$ where $\varphi_1 = \varphi_0 p$. This homomorphism is in $\ker \delta$, and we send it to its quotient class $[\varphi_1]$ in $H^n(C; G)$. We then apply h: we first choose a representative element of φ_2 of $[\varphi_1]$, which is a cocycle and thus must vanish on B_n. We then restrict its domain to Z_n; this gives us φ_0 again. We finally look at the induced quotient homomorphism $\bar{\varphi_0}$, which is indeed φ_{-1}.

Thus h must be surjective onto $Hom(H_n(C), G)$. □

It is easily seen that the elements of $\ker h$ are the cohomology classes of those homomorphisms that vanish on Z_n. These homomorphisms all have cohomology classes because they must vanish on boundaries and hence are cocycles. If every homomorphism that vanishes on Z_n is a coboundary, then these cohomology classes are all $[0]$ and hence $\ker h$ is trivial. In this case $h : H^n(C;G) \xrightarrow{h} Hom(H_n(C), G)$ is injective and is hence an isomorphism.

We have the following split short exact sequence:

$$0 \to \ker h \to H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0 \tag{5.4.2}$$

where $\ker h$ is trivial if and only if every homomorphism that vanishes on Z_n is a coboundary. If $\ker h$ is nontrivial, the nontrivial elements are cohomology classes of homomorphisms that vanish on Z_n but are not coboundaries.

Consider now the short exact sequence:

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0. \tag{5.4.3}$$

The map i_{n-1} is the inclusion map and the map on the right is the quotient map.

Dualize this sequence via $Hom(--, G)$ to obtain

$$0 \leftarrow B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow H_{n-1}(C)^* \leftarrow 0. \tag{5.4.4}$$

This sequence need not be exact, but the dual of a split short exact sequence is always a split short exact sequence, so if (5.4.3) splits then (5.4.4) is split exact.

Let δ' be the natural extension of the map δ to all homorphisms in B^*_{n-1}. If $\varphi_0 \in B^*_{n-1}$ then $\delta'\varphi_0$ is a homomorphism from C_n to G; i.e. it is an element of C_{n-1}. It is not necessarily a coboundary, but it does vanish on Z_n. Furthermore, for each nontrivial class of homomorphisms $[\psi] \in \ker h$ that vanishes on Z_n, we can find homomorphisms ψ in the image of δ' (note that $[\psi]$ being nontrivial implies that such ψ are not coboundaries.)

The map $\delta'\varphi_0$ a coboundary if and only if $\varphi_0 : B_{n-1} \to G$ can be extended to a homomorphism $\varphi : C_{n-1} \to G$. This is the case if and only if it can first be extended to a homomorphism $\varphi_1 : Z_{n-1} \to G$. In other words, $\varphi_0 \in \text{Im } i^*_{n-1}$.

The cokernel of i^*_{n-1} is the quotient $B^*_{n-1}/\text{Im } i^*_{n-1}$. By the arguments above, this cokernel is isomorphic with $\ker h$.

We can then replace (5.4.2) with the new split short exact sequence:

$$0 \to \text{coker } i^*_{n-1} \to H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0. \tag{5.4.5}$$

In order to completely describe the group $H^n(C;G)$ in terms of the groups $H_n(C)$ and G, we wish to describe the term coker i^*_{n-1} simply in terms of the homology groups of C.

In the case that (5.4.4) is exact, coker i^*_{n-1} is zero. This is usually not the case. Sequence (5.4.3) does, however, have a special quality that it is a "free resolution" of $H_{n-1}(C)$.

Definition 5.5. Free Resolution of an Abelian Group.

A **free resolution** of an abelian group H is an exact sequence

$$... \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0 \tag{5.5.1}$$

where each F_n is free. If we dualize a free resolution by looking at the groups of homomorphisms $Hom(--, G)$, we get a dual complex

$$... \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0. \tag{5.5.2}$$

Let us use the temporary notation $H^n(F;G)$ to indicate the cohomology group ker f_{n+1}^*/Im f_n^* of the dual complex.

Lemma 5.6. *Every abelian group H has a free resolution of the form*

$$0 \to F_1 \to F_0 \to H \to 0, \tag{5.6.1}$$

with $F_i = 0$ for $i > 1$.

Proof. We obtain this free resolution as follows. Choose any set of generators for H and let F be a free abelian group with basis in one-to-one correspondence with these generators. We then have a surjective homomorphism $f_0 : F_0 \to H$ sending the basis elements to the chosen generators. The kernel of f_0 is free, being a subroup of a free abelian group; we let F_1 be this kernel with $f_1 : F_1 \to F_0$ as the inclusion map. Take $F_i = 0$ for $i > 1$. We leave it to the reader to check that the sequence is exact. □

In this case $H^n(F;G) = 0$ for $n > 1$.

Lemma 5.7. *For any two free resolutions F and F' of a group H, there are isomorphisms $H^n(F;G) \cong H^n(F';G)$ for all n.*

We omit the proof here.

Corollary 5.8. *For any free resolution F of a group H, $H^n(F;G) = 0$ for $n > 1$.*

This follows immediately from the above two lemmas. $H^1(F;G)$ is the only interesting cohomology group, and it is invariant regardless of the free resolution F. This group depends only on H and G, and is denoted as $Ext(H,G)$.

5.9. **The Universal Coefficient Theorem.** We now return to our discussion of cohomology groups.

Recall the short exact sequence

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0 \tag{5.9.1}$$

and its dual

$$0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}(C)^* \leftarrow 0. \tag{5.9.2}$$

The first sequence above is a free resolution F of $H_{n-1}(C)$. coker $i_{n-1}^* = Z_{n-1}^* / \ker i_{n-1}^*$ is then $H^1(F;G)$ by definition. We have seen that $H^1(F;G)$ is independent of the choice of free resolution F and is dependent only on H and G and is denoted $Ext(H,G)$. In this case the group H is the group of homomorphisms $H_{n-1}(C)$. We have that

$$\text{coker } i^*_{n-1} = Ext(H_{n-1}(C), G). \tag{5.9.3}$$

Replacing terms in the split short exact sequence (5.3.5) gives us the remarkable **Universal Coefficient Theorem for Cohomology**, whose statement is as follows.

Theorem 5.10. *If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split short exact sequences*

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0 \tag{5.10.1}$$

Proposition 5.11. *In order to utilize this theorem we note some properties about the groups $Ext(H,G)$.*

5.11.1. $Ext(H \oplus H', G) \cong Ext(H,G) \oplus Ext(H', G)$

5.11.2. $Ext(H,G) = 0$ if H is free

5.11.3. $Ext(\mathbb{Z}_n, G) \cong G/nG$

Proof. We can use a direct sum of free resolutions for H and H' as a free resolution for $H \oplus H'$. (5.10.1) follows directly when we look at cohomology groups. If H is free, then $0 \to H \to H \to 0$ is a free resolution of H, so we get (5.10.2) $H^1(F,G) = 0$.

To find the third property we use the free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$. This gives a dual sequence $0 \leftarrow Hom(\mathbb{Z}, G) \xleftarrow{n} Hom(\mathbb{Z}, G) \leftarrow Hom(\mathbb{Z}_n, G) \leftarrow 0$. The groups $Hom(\mathbb{Z}, G)$ here are isomorphic to G, so the image of $Hom(\mathbb{Z}, G)$ under the homomorphism n is isomorphic to nG and $Ext(\mathbb{Z}_n, G) \cong G/nG$. □

Properties (5.10.1) and (5.10.2) imply that $Ext(H,G) \cong Ext(T,G)$, where T is the torsion subgroup of H, if the free part of H is finitely generated. Properties (5.10.1) and (5.10.3) imply that $Ext(T, \mathbb{Z}) \cong T$ if T is finitely generated. So we have that $Ext(H, \mathbb{Z})$ is isomorphic to the torsion subgroup of H if H is finitely generated.

We can also see that $Hom(H, \mathbb{Z})$ is isomorphic to the free part of H if H is finitely generated, so the splitting of the sequence in the Universal Coefficient Theorem gives us the following corollary:

Corollary 5.12. *If the homology groups $H_n(C)$ and $H_{n-1}(C)$ of a chain complex C of free abelian groups are finitely generated, with torsion subgroups $T_n \subset H_n(C)$ and $T_{n-1} \subset H_{n-1}(C)$, then $H^n(C;\mathbb{Z}) \cong T_{n-1} \oplus (H_n(C)/T_n)$.*

To conclude this section we have one more powerful corollary:

Corollary 5.13. *If a chain map between chain complexes of free abelian groups induces an isomorphism on homology groups, then it induces an isomorphism on cohomology groups with any coefficient group G.*

6. Cohomology Groups of Spaces

We now finally turn our attention to the cohomology groups of spaces.

Definition 6.1. Given a space X and an abelian group G, define $C^n(X;G)$, the group of singular n-cochains of X with coefficients in G, to be the dual group $Hom(C_n(X),G)$ of the singular cochain group $C_n(X)$.

Remark 6.2. A homomorphism $\varphi \in C^n(X,G)$ assigns to each singular n-simplex $\sigma : \Delta^n \to X$ a value $\varphi(\sigma) \in G$. Since the singular n-simplices are a basis for $C_n(X)$, these values can be chosen arbitrarily, and define the entire homomorphism φ. n-cochains are equivalent to functions from singular n-simplices to G.

Definition 6.3. Define the coboundary map $\delta : C^{n-1}(X;G) \to C^n(X;G)$ to be the dual of the boundary map $\partial : C_n(X) \to C_{n-1}(X)$. Define cohomology groups $H^n(X;G)$ with coefficients in G to be the quotient groups $\ker\delta/\operatorname{Im}\delta$ at $C^n(X;G)$ in the cochain complex.

Remark 6.4. For a singlar $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$, we have

$$\delta\varphi(\sigma) = \varphi(\partial\sigma) = \varphi(\sum_{i=1}^{n}(-1)^i\sigma \mid [v_0, ..., \bar{v_i}, ..., v_{n+1}]) = \sum_{i=1}^{n}(-1)^i\varphi(\sigma \mid [v_0, ..., \bar{v_i}, ..., v_{n+1}]).$$

We now have extended our definition of cohomology groups of chain complexes naturally to a definition of cohomology groups of spaces. Since the chain groups $C_n(X)$ are free, we can extend the theorems for cohomology groups of chain complexes all naturally follow to cohomology groups of chain complexes of spaces. The Universal Coefficient Theorem becomes:

Theorem 6.5. *The cohomology groups $H^n(X;G)$ of a space X with coefficients in G are described by split short exact sequences:*

$$0 \to Ext(H_{n-1}(X),G) \to H^n(X;G) \to Hom(H_n(X),G) \to 0 \tag{6.5.1}$$

Here are a few examples of how the Universal Coefficient Theorem can be easily applied to evaluate cohomology groups of familiar spaces.

Example 6.6. We have seen that the 0th homology group of $\mathbb{P}^2$ is $H_0(\mathbb{P}^2) \cong \mathbb{Z}$, and the first homology group of $\mathbb{P}^2$ is $H_1(\mathbb{P}^2) \cong \mathbb{Z}_2$. The torsion subgroup of $\mathbb{Z}$ is isomorphic to the trivial group, and thus $Ext(\mathbb{Z},G) = Ext(0,G)$ is the trivial group. Therefore the first cohomology group $H^1(\mathbb{P};G)$ is isomorphic to the group of homomorphisms from $\mathbb{Z}_2$ to G, which is trivial if G is free and otherwise is a direct sum of copies of $\mathbb{Z}_2$. The second homology group of $\mathbb{P}^2$ is trivial, and $Ext(\mathbb{Z}_2,G)$ is isomorphic to $G/2G$, so the second cohomology group $H^2(\mathbb{P};G)$ is isomorphic to $G/2G$.

Example 6.7. Both the 0th and first homology groups of S_1 are isomorphic to $\mathbb{Z}$, and the higher homology groups of S_1 are all isomorphic to the trivial group. Thus the first cohomology group $H^1(S_1;G)$ is isomorphic to the group of homomorphisms from $\mathbb{Z}$ to G, which is isomorphic to G.

Example 6.8. The first homology group of the torus T_1 is $H_1(T_1) \cong \mathbb{Z} \oplus \mathbb{Z}$, and the 0th and second homology groups are each isomorphic to $\mathbb{Z}$. The first cohomology group $H^1(T_1;G)$ is isomorphic to the group of homomorphisms from $\mathbb{Z} \oplus \mathbb{Z}$ to the group G, which is trivial if G

is a torsion group and otherwise is a direct sum of copies of $G \oplus G$. The second cohomology group $H^1(T_1; G)$ is isomorphic to the group of homomorphisms from $\mathbb{Z}$ to the group G, which is isomorphic to G.

In fact, the 0th and first cohomology groups of any space with coefficients in a group G are isomorphic to $Hom(H_0(X), G)$ and $Hom(H_1(X), G)$, respectively, because there is no Ext term when calculating $H^0(X; G)$ and $Ext(H_0(X), G)$ always is trivial because $H_0(X)$ is free.

Calculation of $Ext(H_{n-1}(X), G)$ is extremely easy for most spaces X using proposition (5.11), because the homology groups are finitely generated. Calculation of the group of homomorphisms from $H_n(X)$ to G is also usually fairly straightforward, so the Universal Coefficient Theorem makes calculation of the nth cohomology group $H^n(X; G)$ a trivial task in most circumstances. Corollary 5.12 allows one to completely describe the nth cohomology groups of a space X with coefficients in $\mathbb{Z}$.

If X is path-connected, $H_1(X)$ is the abelianization of the fundamental group $\pi_1(X)$, and thus because G is abelian, $Hom(H_1(X), G)$ and $Hom(\pi_1(X), G)$ are isomorphic.

7. The Long Exact Sequence of a Pair

We conclude this paper with a discussion of relative groups in homology and cohomology.

Definition 7.1. Relative Groups. Given a space X and a subspace $A \subset X$, we call (X, A) a **pair**. Let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. We call this the nth **relative chain group** of X and A.

We have a boundary map defined, $\partial : C_n(X) \to C_{n-1}(X)$. We define the boundary map on $C_n(A)$ to be the restriction of this map to the n-simplices in A, and the image of this map is clearly in $C_{n-1}(A)$ since subsimplices of n-simplices in A are all also simplices in A. We can then define the boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$ as the induced quotient map.

The relation $\partial\partial$ holds here. The nth relative homology group $H_n(X, A)$ for a pair is then defined as $\ker \partial / \operatorname{Im} \partial$.

We now wish to define the relative cohomology groups of a pair.

We have a short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0 \tag{7.1.1}$$

where i is the inclusion map and j is the quotient map. We dualize this sequence via $Hom(--, G)$ as usual and get the sequence

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0. \tag{7.1.2}$$

Claim 7.2. The sequence (7.1.2) is exact.

Proof. It is trivial that the sequence is exact at $C^n(X, A; G)$.

The dual map i^* restricts a cochain on X to a cochain on A. For a function from singular n-simplices in X to G (recall that by Remark 6.2, this function identifies with a homomorphism in $C^n(X)$), the image of this function under i^* is obtained by restricting the domain of the function to singular n-simplices in A (which then identifies with the image homomorphism in $C^n(A)$). Any homomorphism in $C^n(A)$ is identified with a function from singular n-simplices in A to G, and this function can be extended to be defined on all singular n-simplices in X, arbitrarily, and then this extended function identifies with a homomorphism in $C^n(X)$ whose image is the homomorphism in A. Thus i^* is surjective. The sequence is exact at $C^n(A;G)$.

The kernel of i^* consists of all singular cochains in $C^n(X)$ that take the value 0 on singular n-simplices in $A \subset X$. These cochains are identified with homomorphisms in $C^n(X, A; G)$, the set of all homomorphisms from $C_n(X, A) = C_n(X)/C_n(A)$ to G. The kernel is then exactly the image $j^*(C^n(X, A; G))$. Thus the sequence is exact at $C^n(X;G)$. □

We have identified the group $C^n(X, G)$ with the set of functions from singular n-simplices in X to G, and have identified the group $C^n(X, G)$ with the set of functions from singular n-simplices in X to G. The quotient group $C_n(X, A)$ is generated by the singular n-simplices with image not contained in A, so we can view $C^n(X, A; G)$ as identified with functions from singular n-simplices in X that vanish on n-simplices with image completely contained in A.

We have boundary maps defined, $\partial : C_n(X) \to C_{n-1}(X)$ and $\partial : C_n(A) \to C_{n-1}(A)$. We have dual coboundary maps δ defined by applying $Hom(--, G)$. We can define the relative coboundary map, $\delta : C^{n-1}(X, A; G) \to C^n(X, A; G)$ as the map induced by the restrictions of the maps δ. This defines the relative cohomology groups $H^n(X, A; G)$.

Note that the relative coboundary map δ is not the same as the dual of the relative boundary map $\partial : C^n(X, A) \to C_{n-1}(X, A)$.

Because the maps i and j commute with ∂, the maps i^* and j^* commute with δ, and the sequence (7.1.2) becomes a short exact sequence of cochain complexes.

(7.2.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \xrightarrow{\delta} & C^{n-1}(X,A;G) & \xrightarrow{\delta} & C^n(X,A;G) & \xrightarrow{\delta} & C^{n+1}(X,A;G) & \xrightarrow{\delta} & \ldots \\
 & & \downarrow j^* & & \downarrow j^* & & \downarrow j^* & & \\
\ldots & \xrightarrow{\delta} & C^{n-1}(X;G) & \xrightarrow{\delta} & C^n(X;G) & \xrightarrow{\delta} & C^{n+1}(X;G) & \xrightarrow{\delta} & \ldots \\
 & & \downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \\
\ldots & \xrightarrow{\delta} & C^{n-1}(A;G) & \xrightarrow{\delta} & C^n(A;G) & \xrightarrow{\delta} & C^{n+1}(A;G) & \xrightarrow{\delta} & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

As we know from Section 4 of this paper, we can extend this to a long exact sequence of cohomology groups.

(7.2.2)
$$\ldots \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{j^*_*} H^n(X; G) \xrightarrow{i^*_*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \xrightarrow{j^*_*} \ldots$$

Similarly, the short exact sequence (7.1.1) becomes a short exact sequence of chain complexes and we have a long exact sequence of homology groups.

$$... \xrightarrow{j_*} H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} ... \tag{7.2.3}$$

There is a duality relationship between the connecting homomorphisms $\delta : H^n(A;G) \to H^{n+1}(X,A;G)$ and $\partial : H_{n+1}(X,A) \to H_n(A)$.

This dual relationship is expressed by the following commutative diagram:

$$\begin{array}{ccc} H^n(A;G) & \xrightarrow{\delta} & H^{n+1}(X,A;G) \\ \downarrow h & & \downarrow h \\ Hom(H_n(A),G) & \xrightarrow{\partial^*} & Hom(H_{n+1}(X,A),G). \end{array} \tag{7.2.4}$$

We leave it as an exercise to check that $h\delta = \partial^* h$.

References

1] Hatcher, Allen. *Algebraic Topology.* Cambridge University Press, 2002. Chapters 2 and 3.

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