

1. Categories

Note Title

2/1/2010

"Homotopy category" [May: Chapter 2]
"Representable functor"

Need to discuss notion of ctgy.

Def A category \mathcal{C} consists of

(1) a collection $\text{Ob } \mathcal{C}$

(2) for any pair $x, y \in \text{Ob } \mathcal{C}$

a set of morphisms $\text{Map}_{\mathcal{C}}(x, y)$

(3) for each $x \in \text{Ob } \mathcal{C}$, an identity morphism

$1_x \in \text{Map}_{\mathcal{C}}(x, x)$

(4) $x, y, z \in \text{Ob } \mathcal{C}$, a composition map:

$\text{Map}_{\mathcal{C}}(y, z) \times \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{C}}(x, z)$

$(f, g) \longmapsto f \circ g$

Satisfying:

• (Identity) $f \in \text{Map}_{\mathcal{C}}(x, y)$

$$1_y \circ f = f \circ 1_x = f$$

• (associativity)

for composable f, g, h , $(f \circ g) \circ h$

$f \circ (g \circ h)$

Examples

Sets : $Ob = \text{sets}$
morph = set maps

Gp : $Ob = \text{gps}$
morph = gp homomorphisms

Ab : $Ob = \text{ab gps}$
morph = homomorphisms

Mod_R : $Ob = R\text{-modules}$ ($R = \text{ring}$)
morph = R -mod maps

Top : $Ob = \text{top'l spaces}$
morph = cts maps

If \mathcal{C} is a sp, can form a cat

$$\underline{\mathcal{C}} \quad Ob \underline{\mathcal{C}} = *$$

$$Map_{\underline{\mathcal{C}}}(*, *) = \mathcal{C}$$

If \mathcal{C} is a cat, $f: x \rightarrow y$ is an iso

if \exists morphism $f^{-1}: y \rightarrow x$, s.t. $ff^{-1} = 1_y$
 $f^{-1}f = 1_x$

"city of categories?"

Morphisms between cuts: Functors

Def: Let \mathcal{C}, \mathcal{D} be cuts

a ^(contravariant) (covariant) functor

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is a map

$$F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$$

together with set maps

$$F: \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{D}}(F(x), F(y))$$

satisfying $\text{Map}_{\mathcal{D}}(F(y), F(x))$

$$\bullet F(1_x) = 1_{F(x)}$$

$$\bullet F(f \circ g) = F(f) \circ F(g)$$

$$F(g) \circ F(f)$$

$\mathcal{C} = \text{cut}$

$\mathcal{C}^{\text{op}} = \text{opposite cut}$

$$\text{ob } \mathcal{C}^{\text{op}} = \text{ob } \mathcal{C}$$

$$\text{Map}_{\mathcal{C}^{\text{op}}}(y, x) = \text{Map}_{\mathcal{C}}(x, y)$$

There is a bijective correspondence

$$\{\text{contravariant functors } \mathcal{C} \rightarrow \mathcal{D}\}$$



$$\{\text{covariant functors } \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}\}$$

Examples

$$U: \text{Grps} \longrightarrow \text{Sets} \quad \text{forgetful functor}$$

$U(G) = \text{underlying set of } G$

$$F: \text{Sets} \longrightarrow \text{Grps} \quad \text{free gp functor}$$

$F(S) = \text{free gp generated by } S$

$A \in \text{Ab}$

$$\text{Hom}(A, -): \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(A, B)$$

covariant functor (why?)

$$\text{Hom}(-, A): \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(B, A)$$

contravariant functor

$$\text{Hom}(-, -): \text{Ab}^{\text{op}} \times \text{Ab} \longrightarrow \text{Ab}$$

\uparrow product cat.

$$\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$$

$$\text{Map}_{\mathcal{C} \times \mathcal{D}}((x, x'), (y, y')) = \text{Map}_{\mathcal{C}}(x, y) \times \text{Map}_{\mathcal{D}}(x', y')$$

$$\mathcal{B}_1 : \text{Top}_* \longrightarrow \text{Grp}$$

category of pointed spaces
and based pointed maps

$$(X, x) \longmapsto \mathcal{B}_1(X, x)$$

$$H^k(-; \mathbb{R}) : \text{Top} \longrightarrow \text{Mod}_{\mathbb{R}}$$

$$X \longmapsto H^k(X; \mathbb{R})$$

Functors are maps between categories

Natural transformations are maps between functors

Def $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ (Covariant) functors

A natural transformation

is a collection of morphisms in \mathcal{D}

$$\eta_x : F(x) \longrightarrow G(x)$$

$\forall x \in \text{Ob } \mathcal{C}$

that satisfy: \forall morphism $f : x \rightarrow y$ in \mathcal{C}

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \eta_x \downarrow & & \downarrow \eta_y \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

e.g. Suppose $A, A' \in \mathcal{A}$

$$\alpha: A \rightarrow A' \quad \text{isomorphism}$$

get natural transformation

$$\alpha_*: \text{Hom}(-, A) \rightarrow \text{Hom}(-, A')$$

given B

$$(\alpha_*)_B: \text{Hom}(B, A) \longrightarrow \text{Hom}(B, A')$$

$$f \longmapsto \alpha \circ f$$

HW: "adjoint functors"

$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

e.g.

$$F: \text{Sets} \rightleftarrows \text{Grp} : U$$

$$\text{Map}_{\text{Grp}}(F(S), G) \cong \text{Map}_{\text{sets}}(S, UG)$$

$$R_x - : \text{Sets} \rightleftarrows \text{Sets} : \text{Map}(R, -)$$

$$\text{Map}_{\text{set}}(R \times S, T) \cong \text{Map}_{\text{set}}(S, \text{Map}(R, T))$$

$$f \longmapsto (s \longmapsto (r \longmapsto f(r, s)))$$

$$A \otimes - : \text{Ab} \rightleftarrows \text{Ab} : \text{Hom}(A, -)$$

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

"tensor product is dual to hom"
importance of \otimes

Products, coproducts, limits, colimits

$\mathcal{C} = \text{cat}$, $I = \text{small category}$

$$\mathcal{C}^I = \text{caty} \begin{cases} \text{objects} = \text{Functs } I \rightarrow \mathcal{C} \\ \text{Morphs} = \text{natural transformations} \end{cases}$$

category of " I -shaped diagrams
in \mathcal{C} "

eg, $I = \underline{G}$

Discussion: What is

Set I

ie, $\text{Map}(i,j) = \begin{cases} \neq, & i \leq j \\ \emptyset, & \text{otherwise} \end{cases}$

$$I = \{ 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \}$$

$\in I$ objects, $C(0) \leftarrow C(1) \leftarrow C(2) \leftarrow \dots$

Limits

Def:

Let $X: I \rightarrow \mathcal{C}$ be an I -shaped diagram

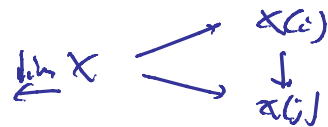
(if it exists)

$$\varprojlim X = \varprojlim_{i \in I} X(i)$$

satisfies the following universal property

There exist maps $\varprojlim X \xrightarrow{\pi_i} X(i)$

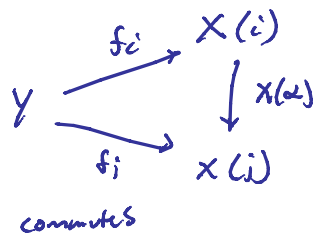
s.t. $\forall \alpha: i \rightarrow j$



Given $Y \in \mathcal{C}$, $f_i: Y \rightarrow X(i)$

s.t.

$\forall \alpha: i \rightarrow j$
in I



$$\exists! \quad f : Y \rightarrow \varprojlim X$$

$$\text{S.t.} \quad \begin{array}{ccc} Y & \longrightarrow & \varprojlim X \\ & \searrow f_i & \downarrow \pi_i \\ & & X(i) \end{array} \quad \begin{array}{l} \text{(commute)} \\ \forall i \end{array}$$

Examples of common \mathcal{I} 's

$$\text{Ob } \mathcal{I} = \{1, 2\} \quad \begin{array}{l} \text{no non-identity} \\ \text{morphisms} \end{array}$$

$$\text{Ob } \mathcal{C}^{\mathcal{I}} = \left\{ (X_1, X_2) \mid X_1, X_2 \in \mathcal{C} \right\}$$

Universal property:

$$\begin{array}{ccc} & & X_1 \\ & \nearrow f_1 & \uparrow \pi_1 \\ Y & \dashrightarrow \exists! & \varprojlim (X_1, X_2) \\ & \searrow f_2 & \downarrow \pi_2 \\ & & X_2 \end{array}$$

e.g. $\mathcal{C} = \text{Sets}$

Q: what is $\varprojlim (X_1, X_2)$?

In general we denote

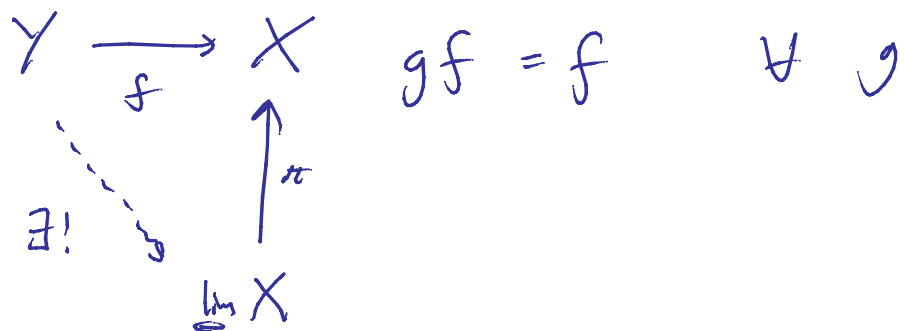
$$\varprojlim (X_1, X_2) := X_1 \times X_2$$

called the ^(categorical) product in \mathcal{C}

e.g. $\mathcal{I} = \underline{G}$, $\mathcal{C} = \text{Sets}$

$$X \in \text{Sets}^{\underline{G}}$$

$$X \supset G$$



What is $\varprojlim X$?

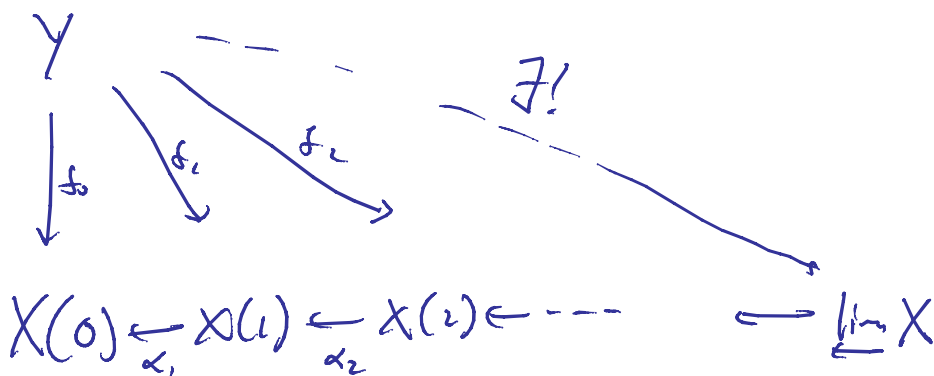
Another important example

$$I = (0 \leftarrow 1 \leftarrow 2 \leftarrow \dots)$$

$$X \in \mathcal{C}^I$$

$$\varprojlim X$$

is called
the projective
limit.



e.g. $\mathcal{C} = \text{sets}$

$$\varprojlim X = \left\{ (x_i)_{i=0}^{\infty} \mid \alpha_i(x_i) = x_{i-1} \right\}$$

e.g.

$$\varprojlim \left(\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \dots \right)$$

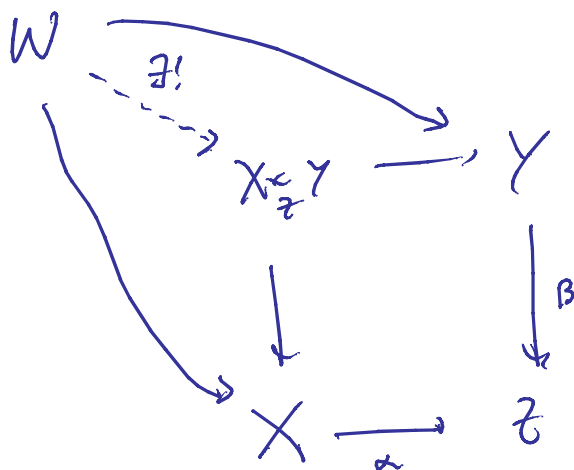
$\cong \mathbb{Z}_p^{\wedge}$

Yet another common example
pull back

$$I = \left\{ \begin{array}{c} z \\ \downarrow \\ 1 \rightarrow 3 \end{array} \right\}$$

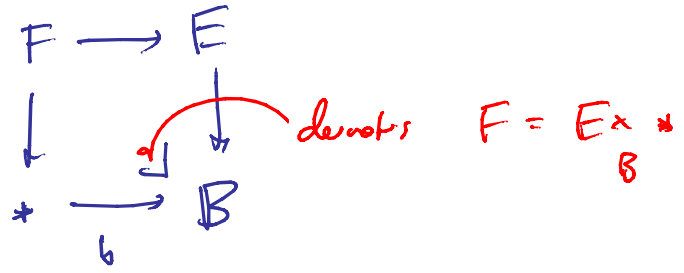
$$e^I = \left\{ \begin{array}{c} Y \\ \downarrow \\ X \rightarrow Z \end{array} \right\}$$

$\lim_{\leftarrow} \left(\begin{array}{c} Y \\ \downarrow \\ X \rightarrow Z \end{array} \right)$ is denoted $X \times_Z Y$



e.g. in Top

$$X \times_Z Y = \left\{ (x, y) \in X \times Y \mid \alpha(x) = \beta(y) \right\} \subset_{\text{subspace}} X \times Y$$



Comits

Same but dual