1. Categories

Note Title 2/1/2010

"Homotopy category" [May: Chapter 2]

"Representable functor"

Need to discuss notion of ctgy.

<u>Def</u> A <u>category</u> $\mathcal{C}$ consists of

(1) a collection $\mathrm{ob}\,\mathcal{C}$

(2) for any pair $x, y \in \mathrm{Ob}\,\mathcal{C}$

a set of morphisms $\mathrm{Map}_{\mathcal{C}}(x, y)$

(3) for each $x \in \mathrm{Ob}\,\mathcal{C}$, an identity morphism

$1_x \in \mathrm{Map}_{\mathcal{C}}(x, x)$

(4) $x, y, z \in \mathrm{ob}\,\mathcal{C}$, a composition map:

$$\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x, z)$$

$$(f, g) \longmapsto f \circ g$$

<u>Satisfying</u>:

- (Identity) $f \in \mathrm{Map}_{\mathcal{C}}(x, y)$

$$1_y \circ f = f \circ 1_x = f$$

- (associativity) for composable f, g, h, $(f \circ g) \circ h = f \circ (g \circ h)$

Examples

Sets : ob = sets

morph = set maps

Gp : ob = gps

morph = gp homomorphisms

Ab : ob = ab gps

morph = homomorphisms

Mod_R : ob = R-modules (R = ring)

morph = R-mod homs

Top : ob = top'l spaces

morph = cts maps

If G is a gp, can form a cat

$\underline{G}$ $\quad$ ob $\underline{G} = *$

$Map_{\underline{G}}(*,*) = G$

If $\mathcal{C}$ is a cat, $f : x \to y$ is an iso

if $\exists$ morphism $f^{-1} : y \to x$, s.t. $ff^{-1} = 1_y$

$f^{-1}f = 1_x$

"Ctgy of categories?"

Morphisms between cats: Functors

<u>Def:</u> Let $\mathcal{C}, \mathcal{D}$ be cats

(contravariant)
a (covariant) functor

$$F: \mathcal{C} \longrightarrow \mathcal{D}$$

is a map

$$F: \mathrm{ob}\,\mathcal{C} \longrightarrow \mathrm{ob}\,\mathcal{D}$$

together with set maps

$$F: \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$$

$\mathrm{Map}_{\mathcal{D}}(F(y), F(x))$

Satisfying

- $F(1_x) = 1_{F(x)}$
- $F(f \circ g) = F(f) \circ F(g)$

$F(g) \circ F(f)$

$\mathcal{C}$ = cat

$\mathcal{C}^{op}$ = opposite cat:

$$\mathrm{ob}\,\mathcal{C}^{op} = \mathrm{ob}\,\mathcal{C}$$

$$\mathrm{Map}_{\mathcal{C}^{op}}(x, y) = \mathrm{Map}_{\mathcal{C}}(y, x)$$

There is a bijective correspondence

$$\{\text{contravariant functors } \mathcal{C} \to \mathcal{D}\}$$
$$\updownarrow$$
$$\{\text{covariant functors } \mathcal{C}^{op} \to \mathcal{D}\}$$

<u>Examples</u>

$\mathcal{U}$: Gps $\longrightarrow$ Sets forgetful functor

$\mathcal{U}(G)$ = underlying set of G

F : Sets $\longrightarrow$ Gps free gp functor

$F(S)$ = free gp generated by S

$A \in$ Ab

$$\mathrm{Hom}(A, -) : \mathrm{Ab} \longrightarrow \mathrm{Ab}$$
$$B \longmapsto \mathrm{Hom}(A, B)$$

Covariant functor (why?)

$$\mathrm{Hom}(-, A) : \mathrm{Ab} \longrightarrow \mathrm{Ab}$$
$$B \longmapsto \mathrm{Hom}(B, A)$$

contravariant functor

$$\mathrm{Hom}(-,-) : \mathrm{Ab}^{op} \times \mathrm{Ab} \longrightarrow \mathrm{Ab}$$

product cat.

$$\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$$
$$\mathrm{Map}_{\mathcal{C}\times\mathcal{D}}((x, x'), (y, y')) = \mathrm{Map}_{\mathcal{C}}(x, y) \times \mathrm{Map}_{\mathcal{D}}(x', y')$$

$$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$$

category of pointed spaces and bspt preserving maps

$$(X, x_0) \longmapsto \pi_1(X, x_0)$$

$$H^k(-;R) : \mathrm{Top} \longrightarrow \mathrm{Mod}_R$$

$$X \longmapsto H^k(X;R)$$

Functors are maps between categories

Natural transformations are maps between functors

Def $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ (covariant) functors

A <u>natural transformation</u>

is a collection of morphisms in $\mathcal{D}$

$$\eta_x : F(x) \longrightarrow G(x)$$

$\forall x \in \mathrm{Ob}\,\mathcal{C}$

that satisfy: $\forall$ morphism $f : x \to y$ in $\mathcal{C}$

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

e.g. Suppose $A, A' \in Ab$

$\alpha : A \longrightarrow A'$ homomorphism

get natural transformation

$$\alpha_* : Hom(-, A) \longrightarrow Hom(-, A')$$

given B

$$(\alpha_*)_B : Hom(B, A) \longrightarrow Hom(B, A')$$

$$f \longmapsto \alpha \circ f$$

HW: "adjoint functors"

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

e.g. $F : \text{Sets} \rightleftarrows \text{Gp} : \mathcal{U}$

$$\text{Map}_{\text{Gp}}(F(S), G) \cong \text{Map}_{\text{Sets}}(S, \mathcal{U}G)$$

$$R\times - : \text{Sets} \rightleftarrows \text{Sets} : \text{Map}(R, -)$$

$$\text{Map}_{\text{Set}}(R\times S, T) \cong \text{Map}_{\text{Set}}(S, \text{Map}(R,T))$$

$$f \longmapsto (s \longmapsto (r \longmapsto f(r,s)))$$

$$A\otimes - : \text{Ab} \rightleftarrows \text{Ab} : \text{Hom}(A, -)$$

$$\text{Hom}(A\otimes B, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

"tensor product is dual to hom"
importance of $\otimes$

Products, coproducts, limits, colimits

$\mathcal{C}$ = cat, I = small category

$\mathcal{C}^I$ = catgy $\begin{cases} \text{objects} = \text{Functors} : I \to \mathcal{C} \\ \text{Morph} = \text{natural transformations} \end{cases}$

category of "I-shaped diagrams in $\mathcal{C}$"

e.g. $\quad I = \underline{G}$

Discussion: what is $\mathrm{Set}^{\underline{G}}$

i.e. $\mathrm{Map}(i,j) = \begin{cases} G, & i=j \\ \emptyset, & \text{o/w} \end{cases}$

$$I = \{ 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \}$$

$\mathcal{C}^I$ objects: $C(0) \leftarrow C(1) \leftarrow C(2) \leftarrow \cdots$

Limits

Def: Let $X : I \to \mathcal{C}$ be an I-shaped diagram

(if it exists)

$\varprojlim X = \varprojlim_{i \in I} X(i)$ satisfies the following universal property

There exist maps $\varprojlim X \xrightarrow{\pi_i} X(i)$

s.t. $\forall\ \alpha : i \to j$

$$\begin{tikzcd} & X(i) \arrow[d] \\ \varprojlim X \arrow[ur] \arrow[r] & X(j) \end{tikzcd}$$

Given $Y \in \mathcal{C}$, $f_i : Y \to X(i)$

s.t. $\forall\ \alpha : i \to j$ in I

$$\begin{tikzcd} & X(i) \arrow[d, "X(\alpha)"] \\ Y \arrow[ur, "f_i"] \arrow[r, "f_j"'] & X(j) \end{tikzcd}$$

commutes

$\exists! \quad f : Y \to \varprojlim X$

s.t.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "f_i"'] & \varprojlim X \arrow[d, "\pi_i"] \\ & X(i) \end{tikzcd}$$

(commutes) $\forall i$

<u>examples of common I's</u>

$\mathrm{Ob}\, I = \{1 \quad 2\}$ no non-identity morphisms

$\mathrm{Ob}\, \mathcal{C}^I = \{(X_1, X_2) \mid X_1, X_2 \in \mathcal{C}\}$

<u>Universal property</u>:

$$\begin{tikzcd} & X_1 \\ Y \arrow[ur, "f_1"] \arrow[r, dashed, "\exists!"'] \arrow[dr, "f_2"'] & \varprojlim (X_1, X_2) \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ & X_2 \end{tikzcd}$$

e.g. $\mathcal{C} = \mathrm{Sets}$

Q: what is $\varprojlim (X_1, X_2)$?

In general we denote

$$\varprojlim (X_1, X_2) := X_1 \times X_2$$

called the (categorical) product in $\mathcal{C}$

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e.g. $I = G$, $\mathcal{C} = \mathrm{Sets}$

$X \in \mathrm{Sets}^G$

$X \circlearrowleft G$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, dashed, "\exists!"'] & X \\ & \varprojlim X \arrow[u, "\pi"'] \end{tikzcd} \qquad gf = f \quad \forall g$$

What is $\varprojlim X$?

Another important example

$$I = \left(0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots\cdots \right)$$

$X \in \mathcal{C}^I$ $\qquad \varprojlim X$ is called the projective limit.

$$\begin{tikzcd}
Y \arrow[d, "f_0"] \arrow[dr, "f_1"] \arrow[drr, "f_2"] \arrow[drrrr, dashed, "\exists !"] & & & & \\
X(0) & X(1) \arrow[l, "\alpha_1"] & X(2) \arrow[l, "\alpha_2"] & \cdots \arrow[l] & \varprojlim X \arrow[l]
\end{tikzcd}$$

e.g. $\mathcal{C} = \text{Sets}$

$$\varprojlim X = \left\{ (x_i)_{i=0}^{\infty} \;\middle|\; \alpha_i(x_i) = x_{i-1} \right\}$$

e.g. $$\varprojlim \left(\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots \right) = \hat{\mathbb{Z}}_p$$

Yet another common example

pull back

$$I = \left\{ \begin{tikzcd} & 2 \arrow[d] \\ 1 \arrow[r] & 3 \end{tikzcd} \right\}$$

$$\mathcal{C}^I = \left\{ \begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right\}$$

$$\varprojlim \left(\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right)$$ is denoted $X \times_Z Y$

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

e.g. in Top

$$X \times_Z Y = \{ (x,y) \in X \times Y \mid \alpha(x) = \beta(y) \} \underset{\text{subspace}}{\subset} X \times Y$$

$$
\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
* \arrow[r, "b"'] & B
\end{tikzcd}
$$

denotes $F = E \times_B *$

Colimits

Same but dual