10 - Brown Representability

Note Title 3/2/2010

Brown Representability Thm

Need to
- construct $K(\pi, n)$
- prove $[X, K(\pi,n)]_* \cong \tilde{H}^n(X;\pi)$ [for X a CW cx]

Def:

pointed connected CW cx's

$$F : (Top_*^{CW_0})^{op} \longrightarrow Sets_*$$

is called an *excisive homotopy functor* if it satisfies the following axioms

homotopy axiom:

$f, g \in Top_*(X, Y)$

$f \simeq g \in [X, Y]_*$

$$\Rightarrow \quad f^* = g^* : F(Y) \longrightarrow F(X)$$

Mayer-Vietoris:
(homotopy sheaf)

$$Z = X \cup_A Y \qquad \begin{array}{l} i : A \hookrightarrow X \\ j : A \hookrightarrow Y \\ \text{CW pairs} \end{array}$$

$x \in F(X)$

$y \in F(Y)$

$$x|_A = y|_A$$

$$\Rightarrow \exists\, z \in F(Z) \qquad \begin{array}{l} z|_X = x \\ z|_Y = y \end{array}$$

Wedge axiom for arbitrary indexing sets:

$$F\left(\bigvee_\alpha X_\alpha\right) \longrightarrow \prod_\alpha F(X_\alpha)$$

(i) empty wedge $= *$

empty product $= *$

$\Rightarrow \quad F(*) = *$

(ii) $A \xrightarrow[i]{} X$ cw pair

$$C(i) \simeq X/A$$

$$C(i) = X \cup_A C(A)$$

Consider

$$\begin{array}{ccccc} F(A) & \longleftarrow & F(X) & \longleftarrow & F(C(i)) \\ & & & & \wr\!\!\upharpoonright \leftarrow \text{htpy axiom} \\ & & & & F(X/A) \end{array}$$

Suppose $x \in F(X)$

$x|_A = *$ $\quad \overset{}{\underset{\text{M-V}}{\Longrightarrow}} \quad$ $\exists\ z \in C(i)$ s.t. $z|_X = x$

$F(C(A)) = *$

i.e. this seq is exact

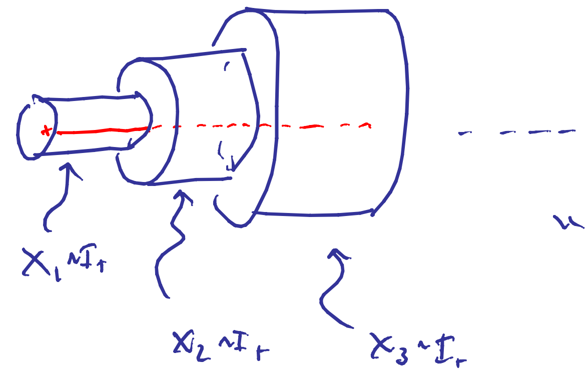
11 - pf of Brown Representability, $K(\pi,n)$, BG

Note Title 3/4/2010

(iii) Wedge axiom: (subcomplexes)

Suppose $X = (X_1 \hookrightarrow X_2 \hookrightarrow \cdots)$

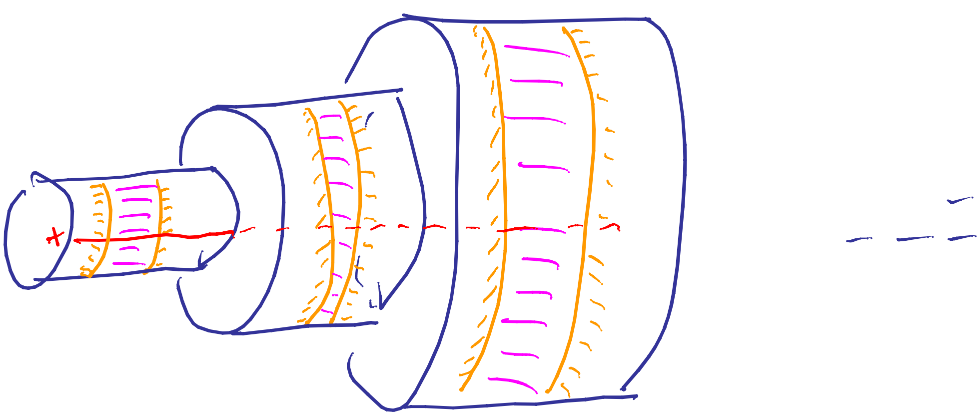
$X \simeq$



" $\mathrm{Tel}(X_i)$ "

called "reduced telescope"

$\mathrm{Tel}(X_i) \simeq X$



$$
\begin{tikzcd}
\bigvee_i X_i \vee \bigvee_i X_i \arrow[r, "\mathrm{Id} \vee f"] \arrow[d, "\mathrm{Id} \vee \mathrm{Id}"'] & \bigvee_{i \geq 1} X_i \arrow[d] \\
\bigvee_{i \geq 1} X_i \arrow[r] & \mathrm{Tel}(X_i)
\end{tikzcd}
$$

$f|_{X_i} : X_i \hookrightarrow X_{i+1} \hookrightarrow \bigvee X_i$

$\longleftarrow$ up to htpy

$x_i \in F(X_i)$ $\qquad$ $x_i|_{X_{i-1}} = x_{i-1}$

<u>Wedge cons</u> $\qquad (x_i)_i \in F(\vee X_i)$

$$(x_1, x_2, \dots) \times (x_2|_{X_1}, x_3|_{X_2}, \dots) \longleftarrow\!\shortmid (x_1, x_2, \dots)$$

$$(x_1, x_2, \dots) \times (x_1, x_2, \dots)$$

$$\uparrow$$

$$(x_1, x_2, \dots)$$

$$\Longrightarrow \quad z \in F(\mathrm{Tel}(X)) \cong F(X) \quad \text{s.t.}$$

$$z|_{X_i} = x_i$$

<u>i.e.</u> $\qquad F(X) \longrightarrow \varprojlim F(X_i)$

is <u>surjective</u> (not iso in general)

E.g. K = pointed CW cx:

$$F(X) = [X, K]_*$$

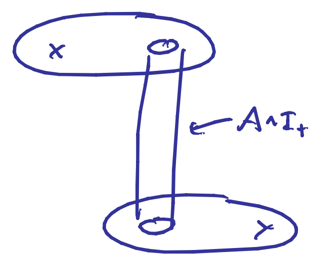
htpy axiom ✓

wedge ✓

MV?

$$Z = X \cup_A Y$$

$$\tilde{Z} = X \cup A \wedge I_+ \cup Y$$



Note $\tilde{Z} \simeq Z$

$f \in F(X)$ H: $f|_A \simeq g|_A$

$g \in F(Y)$

$$X \cup A \wedge I_+ \cup Y \xrightarrow[f \cup H \cup g]{} K$$

restricts to f, g on X and Y.

E.g. $F(X) = \tilde{H}^n(X;\pi)$

(Rmk) So $\tilde{H}^*(X) \twoheadrightarrow \varprojlim H^n(X_i)$ is it iso?

$$\begin{tikzcd}
\bigvee_i X_i \vee \bigvee_i X_i \arrow[r, "\mathrm{Id}\vee f"] \arrow[d, "\mathrm{Id}\vee\mathrm{Id}"'] & \bigvee_{i\geq 1} X_i \arrow[d] \\
\bigvee_{i\geq 1} X_i \arrow[r] & \mathrm{Tel}(X_i)
\end{tikzcd}$$

$\Longrightarrow$ MV:

$$\begin{tikzcd}
 & & & 0 & & 0 \arrow[dl] \\
 & & & & K_n \arrow[ul] \arrow[dll] & \\
\cdots & \prod_i \left(\tilde{H}^n(X_i)\oplus \tilde{H}^n(X_i)\right) \arrow[l] & & \prod_i \left(\tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i)\right) \arrow[ll, "{\begin{bmatrix} 1 & -1 \\ 1 & -f \end{bmatrix}}"'] & & \tilde{H}^n(X) \arrow[ll] \arrow[ul] \\
 & 0 & & & & \\
0 \arrow[r] & C_{n-1} \arrow[u] \arrow[uurrrr, bend right] & & & & \\
 & & \prod_i \left(\tilde{H}^{n-1}(X_i)\oplus \tilde{H}^{n-1}(X_i)\right) \arrow[ul] \arrow[uuurrr, bend left] & & \prod_i \left(\tilde{H}^{n-1}(X_i)\oplus\tilde{H}^{n-1}(X_i)\right) \arrow[ll] & \cdots \arrow[l]
\end{tikzcd}$$

$\begin{bmatrix} 1 & -1 \\ 1 & -f \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$(x,y) \longmapsto (x-y,\, x-f(y))$

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & \prod_i \tilde{H}^n(X_i) \arrow[u] \arrow[rr, "1"] & & \prod_i \tilde{H}^n(X_i) \arrow[u] & 0 \\
K_n \arrow[u] \arrow[r] & \prod_i\left(\tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i)\right) \arrow[u, "\pi_1-\pi_2"] \arrow[rr, "{\begin{bmatrix} 1 & -1 \\ 1 & -f \end{bmatrix}}"'] & & \prod_i \tilde{H}^n(X_i)\oplus\tilde{H}^n(X_i) \arrow[u, "\pi_1"'] \arrow[d, "\iota_2"] \arrow[r] & C_n \arrow[u] \\
K_n' \arrow[u] \arrow[r] & \prod_i \tilde{H}^n(X_i) \arrow[u, "\Delta"', "(\varepsilon,1)"] \arrow[rr, "1-f"'] & & \prod_i \tilde{H}^n(X_i) \arrow[r] & C_n' \arrow[u] \\
 & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

Snake lemma

$$\Longrightarrow \quad K_n' \xrightarrow{\cong} K_n$$

$$C_n' \xrightarrow{\cong} C_n$$

$$K_n' = \varprojlim \tilde{H}^n(X_i)$$

$$C_n' := \varprojlim{}^1 \tilde{H}^n(X_i)$$

We have proven

Prop there is a SES:

$$0 \to \varprojlim{}^1 \tilde{H}^{n-1}(X_i) \to \tilde{H}^n(X) \to \varprojlim \tilde{H}^n(X_i) \to 0$$

Consider $I = (1 \leftarrow 2 \leftarrow \cdots\cdots)$

abelian category

$$Ab^I \xrightarrow{\varprojlim} Ab$$

$\varprojlim$ is NOT exact $\qquad R^1\varprojlim := \varprojlim{}^1$

(HW) (contrast w/ $\varinjlim \tilde{H}_*(X_i) = \tilde{H}_*(X)$)

Note Title 3/9/2010

Thm: (Brown Representability)

Suppose F is an excision htpy functor.

Then $\exists$ pointed, connected CW cx $K = K_F$, unique up to htpy, and $u \in F(K)$

s.t.

$$[X, K]_* \longrightarrow F(X)$$

$$f \longmapsto f^* u$$

is an isomorphism.

Rmk uniqueness of K up to htpy

(i) follows from Yoneda:

$$[-, K]_* \underset{\substack{\text{naturally}\\ \text{iso}}}{\cong} [-, K']_*$$

$\Leftrightarrow$ K isomorphic to K' in $Ho(Top_*^{CW})$

$\Leftrightarrow$ $K \simeq K'$

(iii) Yoneda lemma: F, G htpy excision functors

$$\Longrightarrow \quad Nat(F,G) \cong [K_F, K_G]_*$$

(iv) In fact:

F, G excision htpy functors

$\eta: F \longrightarrow G$ natural transformation

and $\eta: F(S^n) \xrightarrow{\cong} G(S^n)$

$$\begin{array}{ccc} F(S^n) & \xrightarrow[\cong]{} & G(S^n) \\ \wr\| & & \wr\| \\ \pi_n(K_F) & & \pi_n(K_G) \end{array}$$

$\Longrightarrow \quad K_F \longrightarrow K_G$ w.e.

$\Rightarrow$ h.e.

iii any $K(\pi, n)$ represents $\tilde{H}^n(-;\pi)$

Given (K,u) get nat transf

$u \in F(K)$ $\quad [-, K]_* \longrightarrow F(-)$

$f \longmapsto f^* u$

(K,u) is universal if this map is an equivalence.

(pf of Brown Representability)

<u>Def:</u> (K, u) n-universal if

$$\pi_i(K) \longrightarrow F(S^i)$$
$$f \longmapsto f^* u$$

epi for $i \leq n$
trivial kernel for $i < n$

| ∞-universal

(<u>Rmk:</u> $i \geq 1 \Rightarrow F(S^i)$ is a gp
$i > 1 \Rightarrow F(S^i)$ is an ab gp

Since $F(S^i) \times F(S^i) \cong F(S^i \vee S^i) \xrightarrow[\text{pinch}^*]{} F(S^i)$

above is a gp law.)

<u>Lemma:</u> Given (Z, z) ← pointed connected CW cx, $z \in F(Z)$

$\exists$ ∞-universal pair (K, u)

$$(Z, z) \underset{\text{subcomplex}}{\hookrightarrow} (K, u)$$

(pf) $K_1 = Z \vee \bigvee_{x \in F(S^1)} S^1$

$F(K_1) = F(Z) \times \prod_{F(S^1)} F(S^1)$

choose $u_1 = (z, (x)_{x \in F(S^1)})$

$$\pi_1 K_1 \twoheadrightarrow F(S^1) \qquad 1\text{-universal}$$

Inductively assume (K_n, u_n) is n-universal.

$$\ker_n \longrightarrow \pi_n(K_n) \twoheadrightarrow F(S^n)$$

$$\bigvee_{x \in \ker_n} S^n \xrightarrow[\vee x]{} K_n \longrightarrow K_n'$$

__Note:__

$$\prod_{\ker_n} F(S^n) \longleftarrow F(K_n) \longleftarrow F(K_n')$$

$$u_n \longleftarrow\!\!\shortmid \; u_n' \qquad (u_n \in F(K_n),\ u_n' \in F(K_n'))$$

$$K_{n+1} = K_n' \vee \bigvee_{y \in F(S^{n+1})} S^{n+1}$$

$$F(K_{n+1}) = F(K_n') \times \prod_{y \in F(S^{n+1})} F(S^{n+1})$$

$$\in$$

$$u_{n+1} := \left(u_n', (y)_{y \in F(S^{n+1})}\right)$$

K_{n+1} obtained from K_n by attaching $(n+1)$-cells

$$
\begin{tikzcd}
\alpha' \in \pi_i(K_n) \arrow[rr, "{\text{iso } i \le n-1,\ \text{epi } i = n}"] \arrow[dr, "{\text{iso } i<n,\ \text{epi } i=n}"'] & & \pi_i(K_{n+1}) \ni \alpha \arrow[dl, "{\text{epi } i=n;\ \text{iso } i<n,\ \text{epi } i=n+1}"] \\
& F(S^i) &
\end{tikzcd}
$$

Suppose $\alpha \in \pi_n(K_{n+1})$

$$\alpha^* u_{n+1} = 0$$

$$\Rightarrow (\alpha')^* u_n = 0$$

$$\Rightarrow \alpha' \mapsto 0 \text{ in } \pi_n(K_{n+1})$$

$$\Rightarrow \alpha = 0$$

$$F(K_\infty) \to \varprojlim F(K_n)$$

$$u \longmapsto (u_n)$$

∎

<u>lemma</u> Suppose (K, u) ∞-universal

(X, A) CW pair

given f, i, there exists $\tilde{f}$:

$$\begin{tikzcd}
(A, a) \arrow[r, "f"] \arrow[d, "i"'] & (K, u) \\
(X, x) \arrow[ur, dashed, "\tilde{f}"'] &
\end{tikzcd}$$

(pf) WLOG, f is inclusion of a subcx

$$Z := K \cup_A X$$

MV $\Rightarrow$ get $z \in F(Z)$

<u>embed</u> $(Z, z) \hookrightarrow (K', u')$ $\leftarrow$ ∞-universal

$$(K, u) \longrightarrow (Z, z) \longrightarrow (K', u')$$

$$\begin{tikzcd}
\pi_i(K) \arrow[rr] \arrow[dr, "\cong"'] & & \pi_i(K') \arrow[dl, "\cong"] \\
& F(S^i) &
\end{tikzcd}$$

So $K \hookrightarrow K'$ is a w.e.

$\Longrightarrow$ (Compression lemma) K is a deformation retract of K'

$$\begin{tikzcd}
(A,a) \arrow[rr] \arrow[d] & & (K,u) \arrow[r, equal] \arrow[dl] & (K,u) \\
(X,x) \arrow[r, hook] & (Z,z) \arrow[r, hook] & (K',u') \arrow[ur] &
\end{tikzcd}$$

$\square$

(pf of Brown rep)

Need to show ∞-universal $\Longrightarrow$ universal.

Suppose (K,u) ∞-universal

$$\Longrightarrow [X,K]_* \longrightarrow F(X)$$

Is this map an iso?

<u>Surjectivity</u>

$x \in F(X)$:

$$\begin{tikzcd}
(*,0) \arrow[r] \arrow[d] & (K,u) \\
(X,x) \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

✓

Injectivity

$$f, g \in [X, K]_*$$

$$f^*u = g^*u = x$$

$$\begin{tikzcd}
(X \vee X, \ (x, x)) \arrow[r, "f \vee g"] \arrow[d] & (K, u) \\
(X \wedge I_+, \ x) \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

$\square$