

13 - Classifying spaces, continued

Note Title

3/16/2010

G-bundles + free G-CW complexes:

$$\text{Lem } \text{G-} : \text{Top} \rightleftharpoons \text{Top}_G : \text{forget}$$

is an adjoint pair of functors

i.e.

$$\begin{array}{ccc} G \times X & \longrightarrow & Z & G\text{-map} \\ & & \updownarrow & \\ X & \longrightarrow & Z & \text{map} \end{array}$$

Lem:

$$\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$$

principal G-bundle
must be trivial

(Pf)

trivialization



section

$$\begin{array}{ccc} & \nearrow E & \\ G \times D^n & \longrightarrow & D^n \\ & \searrow & \end{array}$$

$$\begin{array}{ccc} & \nearrow E & \\ D^n & \longrightarrow & D^n \\ & \searrow & \end{array}$$

exists an

$$\begin{array}{ccc} * & \xrightarrow{f} & E \\ \downarrow & \searrow & \downarrow \\ D^n & \xrightarrow{\quad} & D^n \end{array}$$

$$\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$$

is a some fibration.



Def: A free G -CW ex.
 is a G -space
 $Z = \varinjlim Z^{[n]}$

$$\begin{array}{ccc} \downarrow G \times S^{n-1} & \longrightarrow & Z^{[n-1]} \\ \downarrow \uparrow & & \downarrow \\ \downarrow G \times D^n & \longrightarrow & Z^{[n]} \end{array}$$

Prop: E free G -module, $B = \text{CW ex.}$
 $\downarrow P$
 B

$\Rightarrow E$ is a free G -CW ex.

(P1)

$$\begin{array}{ccc} \downarrow S^{n-1} & \longrightarrow & B^{[n-1]} \\ \downarrow \uparrow & & \downarrow \\ \downarrow D^n & \xrightarrow{\downarrow \alpha_i} & B^{[n]} \end{array}$$

$\alpha_i^* E|_{B^{[n]}}$ trivial \Rightarrow

$$\begin{array}{ccc} \downarrow G \times S^{n-1} & \longrightarrow & E|_{B^{[n-1]}} \\ \downarrow & & \downarrow \\ \downarrow G \times D^n & \longrightarrow & E|_{B^{[n]}} \end{array}$$

□

E
 $\downarrow p$
 B is pointed if $* \in B, e \in p^{-1}(*)$

$$\begin{array}{ccc} e & \longrightarrow & E \\ & \downarrow & \\ G & \longrightarrow & E \\ & \text{Group} & \end{array}$$

Consider: $F : (\text{Top}_*^{CW})^{op} \rightarrow \text{Set}_*$

$$F(X) = \left\{ \begin{array}{l} \text{is} \\ \text{of} \\ \text{pointed} \end{array} \begin{array}{l} \text{classes} \\ \text{of} \\ \text{principal} \\ \text{bundles} \end{array} \text{ under } (X) \right\}$$

Functoriality

$$\begin{array}{ccc} \mathcal{F}^* E & & \\ \text{"} & & \\ E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathcal{F} : X & \longrightarrow & Y \end{array}$$

pointed w/ trivial G-bundle

Claim: F is a topological functor
 $K_F =: BG$

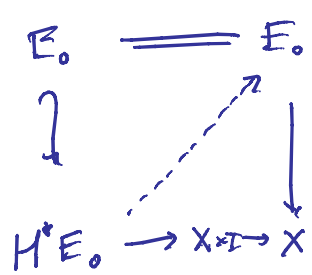
(i) Monotopy $H: \mathcal{F} \simeq \mathcal{F}'$

$$\begin{array}{ccc} H^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & Y \\ & \text{2 pointed} & \\ & \text{topology} & \end{array}$$

Claim:

$$\begin{array}{ccc} H^* E /_{X \times \{0\}} & \simeq & H^* E /_{X \times \{1\}} \\ \text{"} & & \text{"} \\ E_0 & & E_1 \end{array}$$

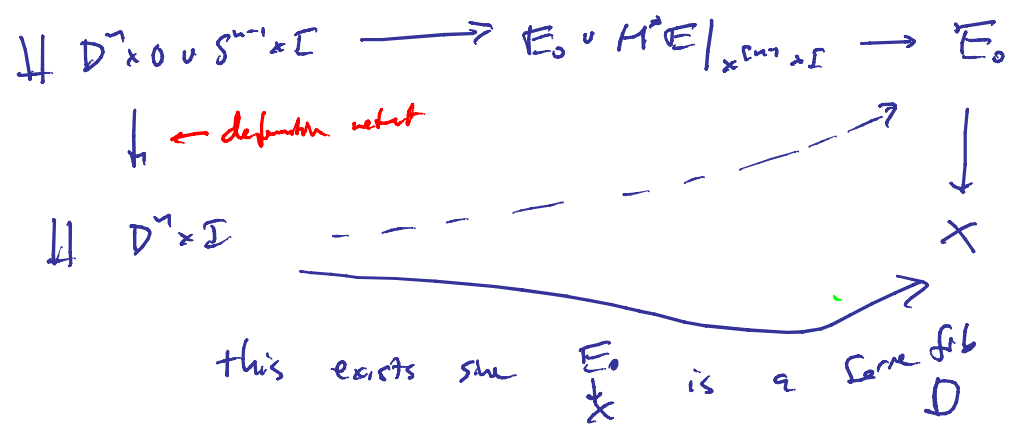
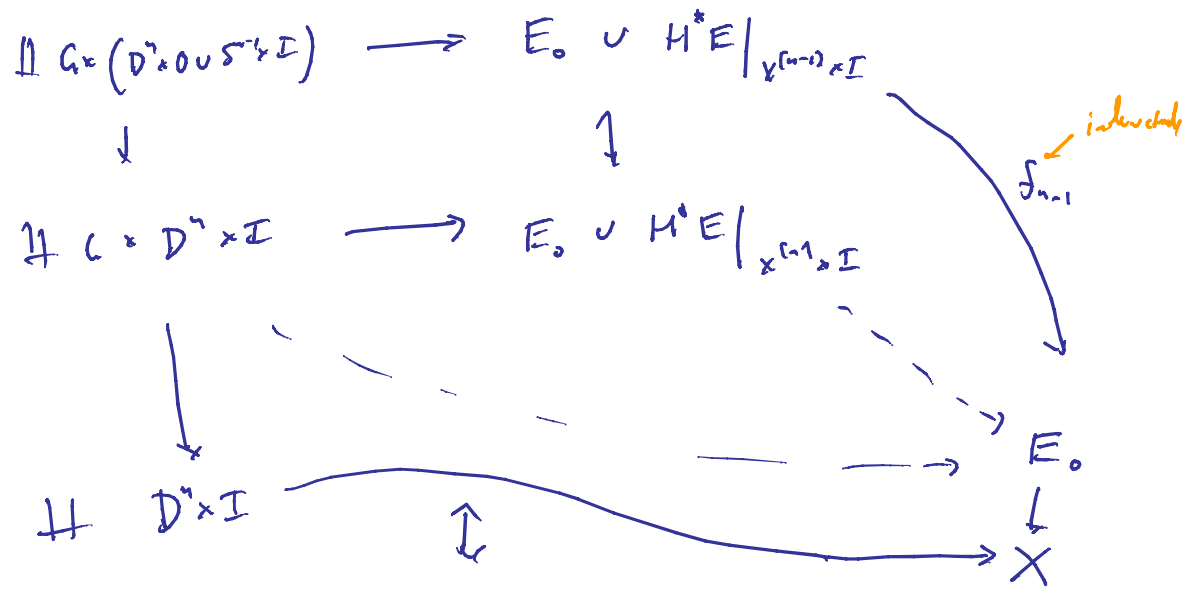
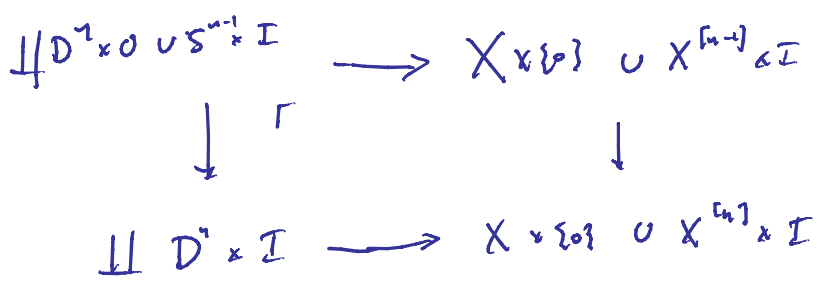
Claim:



(Then $E_1 \rightarrow H^*E_0 \rightarrow E_0$ is the desired iso)

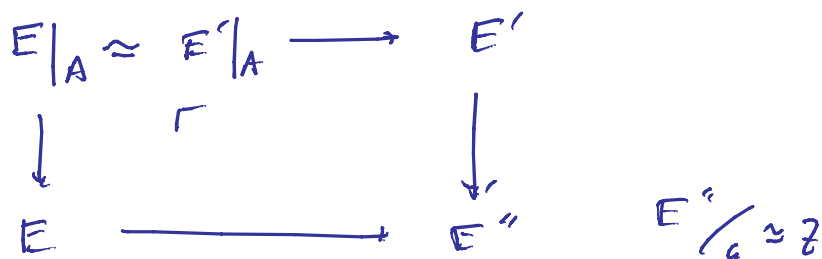
in cat Top_g

Induction

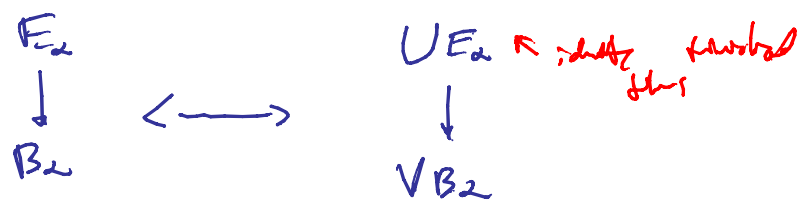


(i) M-V axiom $Z = X \cup_A Y$

$E/x, E'/y \quad E|_A \approx E'|_A$



(ii) Medu



Branched Rep. $\implies \int$ pull in cos B_G

Set. $[x, B_G]_* \cong \{ \text{pointed } G\text{-sets } / x \}$



Thm

(i) If $E \xrightarrow{\pi_2} B$ is a principal G -bundle
 $\forall B = \text{cov } G$
 $\pi_2 E = 0$

$$\Rightarrow B \cong BG \quad (E \cong EG)$$

(ii) $\pi_2 EG = 0$

Consequ: G discrete

$$G \longrightarrow EG \xrightarrow{f \text{ universal cover}} BG$$

\downarrow
 \cong
 $*$

Covering space \rightarrow

$$\Rightarrow \pi_1 BG = G$$

$$(\pi_i BG = 0 \quad i \neq 0)$$

$$\text{So } BG = K(G, 1)$$

$$\left(\begin{array}{l} \text{i.e. } S^1 = B\mathbb{Z} \\ \mathbb{R}P^\infty = B\mathbb{Z}/2 \end{array} \right)$$

acts by moving point upstairs

$$\pi_1 BG \xrightarrow{\quad} [X, BG]_* \rightarrow [X, BG] \rightarrow \pi_0 BG$$

$\pi_0 G \uparrow$

consequence $[X, BG] \cong$ (unpointed)
 principal G -bundles $/ X$

$\left(\Omega BG \cong_{w.e.} G \quad (HW) \right)$

i.e. topological groups "deloop"

example

$$U(1) \hookrightarrow S^{2n+1} \subset \mathbb{C}^{n+1}$$

$$\downarrow$$

$$\mathbb{C}P^{n+1}$$

$$\Rightarrow U(1) \hookrightarrow S^\infty \text{ (contractible)}$$

$$S^\infty = EU(1)$$

$$\downarrow$$

$$\mathbb{C}P^\infty = BU(1)$$

$\hookrightarrow K(\mathbb{Z}, 2)$

Construction of $BO(n)$

$$Frac_n(\mathbb{R}^k) = \{ \text{space of orthogonal } n\text{-frames in } \mathbb{R}^k \}$$

\cap

$$\text{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

$$Frac_n(\mathbb{R}^k) \hookrightarrow O(n)$$

↓

principal $O(n)$ -bundle

$$Gr_n(\mathbb{R}^k) = Frac_n(\mathbb{R}^k) / O(n)$$

\nearrow cur. cov. \nearrow space of n -planes in \mathbb{R}^k

$$Frac_n(\mathbb{R}^\infty) = \varinjlim_k Frac_n(\mathbb{R}^k)$$

\nearrow countable

$$Frac_1(\mathbb{R}^{k-n+1}) \longrightarrow Frac_n(\mathbb{R}^k) \longrightarrow Frac_{n-1}(\mathbb{R}^k)$$

$$\cup$$

$$S^{k-n}$$

$$S^\infty \longrightarrow Frac_n(\mathbb{R}^\infty) \longrightarrow Frac_{n-1}(\mathbb{R}^\infty)$$

$$BO(n) = Gr_n(\mathbb{R}^\infty)$$

Similarly $BU(n) = Gr_n(\mathbb{C}^\infty)$ classifies \mathbb{C} n -plane bundles

$$BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$$

$$\left\{ \begin{array}{l} \text{principal} \\ \text{U(1)-bundles} \\ \text{over } X \end{array} \right\} \cong \left\{ \begin{array}{l} \text{complex line bundles} \\ \text{over } X \end{array} \right\} \cong H^2(X; \mathbb{Z})$$

with

$$\begin{array}{ccccccc} \pi_1(K(\pi, n)) & \rightarrow & [X, K(\pi, n)]_* & \rightarrow & [X, K(\mathbb{Z}, 2)] & \rightarrow & \pi_0(K(\pi, n)) \\ & & \parallel & & \downarrow & & \\ & & \tilde{H}^1(X; \pi) & & H^2(X; \mathbb{Z}) & & \end{array}$$
