13 - Classifying spaces, continued

Note Title 3/16/2010

G-bundles + free G-CW complexes:

Lem $G\times - : Top \rightleftarrows Top_G$: forget

is an adjoint pair of functors

i.e.
$$\begin{array}{ccc} G\times X \longrightarrow Z & & G\text{-map} \\ \updownarrow & & \\ X \longrightarrow Z & & \text{map} \end{array}$$

Lem: $\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$ princ G-bundle must be trivial

(pf) trivialization $\longleftrightarrow$ section

$$\begin{tikzcd} & E \arrow[d] \\ G\times D^n \arrow[ur] \arrow[r] & D^n \end{tikzcd} \longleftrightarrow \begin{tikzcd} & E \arrow[d] \\ D^n \arrow[ur] \arrow[r, equal] & D^n \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \arrow[ur, dashed] \arrow[r, equal] & D^n \end{tikzcd}$$ exists since $\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$ is a Serre fibration.

$\square$

Def: A free G-CW cx. is a G-space

$$Z = \varinjlim Z^{[n]}$$

$$\begin{tikzcd} \coprod G \times S^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Z^{[n-1]} \arrow[d] \\ \coprod G \times D^n \arrow[r] & Z^{[n]} \end{tikzcd}$$

Prop: $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ prin. G-bundle, B = CW cx

$\Rightarrow$ E is a free G-CW cx

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pf:

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B^{[n-1]} \arrow[d] \\ \coprod D^n \arrow[r, "\sqcup \alpha_i"'] & B^{[n]} \end{tikzcd}$$

$\alpha_i^* E|_{B^{[n]}}$ trivial $\Longrightarrow$

$$\begin{tikzcd} \coprod G \times S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E|_{B^{[n-1]}} \arrow[d] \\ \coprod G \times D^n \arrow[r] & E|_{B^{[n]}} \end{tikzcd}$$

$\square$

$\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ is pointed if $* \in B$, $e \in p^{-1}(*)$

$$e \longrightarrow E$$
$$\updownarrow$$
$$G \xrightarrow[\text{G-map}]{} E$$

<u>Consider:</u> $F : (Top_*^{cw})^{op} \longrightarrow Set_*$

$$F(X) = \{\text{iso classes of pointed principal } G\text{-bundles} / X\}$$

pointed w/ trivial G-bdle

<u>Functoriality</u>

$$\begin{tikzcd} f^*E = E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ f : X \arrow[r] & Y \end{tikzcd}$$

<u>Claim:</u> F is a htpy functor

$$K_F =: BG$$

(i) <u>Homotopy</u> $H : f \simeq f'$

$$\begin{tikzcd} H^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r, "\text{pointed htpy}"'] & Y \end{tikzcd}$$

<u>Claim:</u>

$$\underset{E_0}{\underset{\shortparallel}{H^*E|_{X \times \{0\}}}} \cong \underset{E_1}{\underset{\shortparallel}{H^*E|_{X \times \{1\}}}}$$

__Claim:__

$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d, hook] & E_0 \arrow[d] \\
H^*E_0 \arrow[r] \arrow[ur, dashed] & X \times I \arrow[r] & X
\end{tikzcd}$$

in cat Top_q

(Then $E_1 \to H^*E_0 \to E_0$ is the desired iso)

__Induction__

$$\begin{tikzcd}
\coprod D^n \times 0 \cup S^{n-1} \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \times \{0\} \cup X^{[n-1]} \times I \arrow[d] \\
\coprod D^n \times I \arrow[r] & X \times \{0\} \cup X^{[n]} \times I
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod G \times (D^n \times 0 \cup S^{n-1} \times I) \arrow[r] \arrow[d] & E_0 \cup H^*E|_{X^{[n-1]} \times I} \arrow[d, hook] \arrow[ddr, bend left, "f_{n-1}"] \\
\coprod G \times D^n \times I \arrow[r] \arrow[d] \arrow[dr, dashed] & E_0 \cup H^*E|_{X^{[n]} \times I} \arrow[dr, dashed] \\
\coprod D^n \times I \arrow[drr, bend right] & & E_0 \arrow[d] \\
& & X
\end{tikzcd}$$

inductively

$$\begin{tikzcd}
\coprod D^n \times 0 \cup S^{n-1} \times I \arrow[r] \arrow[d, "\leftarrow \text{deformation retract}"'] & E_0 \cup H^*E|_{X^{[n-1]} \times I} \arrow[r] & E_0 \arrow[d] \\
\coprod D^n \times I \arrow[urr, dashed] \arrow[rr, bend right] & & X
\end{tikzcd}$$

this exists since $\begin{matrix} E_0 \\ \downarrow \\ X \end{matrix}$ is a Serre fib

D

(ii) M-V axiom $\qquad Z = X \cup_A Y$

$$E/_X \,,\; E'/_Y \qquad E|_A \cong E'|_A$$

$$\begin{tikzcd}
E|_A \cong E'|_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E' \arrow[d] \\
E \arrow[r] & E''
\end{tikzcd} \qquad E''/_G \simeq Z$$

(iii) wedge

$$\begin{tikzcd}
E_\alpha \arrow[d] \\
B_\alpha
\end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd}
\vee E_\alpha \arrow[d] \\
\vee B_\alpha
\end{tikzcd}$$

identify fibers trivialized

Brown Rep. $\Longrightarrow \exists$ pointed cw cx BG

s.t. $\quad [X, BG]_* \simeq \{\text{pointed } G\text{-bdls} / X\}$

$$BG \xrightarrow[\mathrm{Id}]{} BG \quad \rightsquigarrow \quad \begin{tikzcd}
EG \arrow[d] \\
BG
\end{tikzcd} \qquad \text{"universal } G\text{-bundle"}$$

Thm

(i) If $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a principal G-bundle w/ B = CW cx

$\pi_* E = 0$

$$\Longrightarrow \quad B \simeq BG \qquad (E \simeq EG)$$

(ii) $\pi_* EG = 0$

Consequence: G discrete

$$\begin{array}{ccccc} G & \longrightarrow & EG & \xrightarrow{f} & BG \\ & & \wr\!\wr & & \\ & & * & & \end{array}$$

universal cover

Covering space thy

$$\Longrightarrow \quad \pi_1 BG = G$$

$$(\pi_i BG = 0 \qquad i \neq 0.)$$

So $BG = K(G, 1)$ $\left(\text{i.e. } \begin{array}{l} S^1 = B\mathbb{Z} \\ \mathbb{RP}^\infty = B\mathbb{Z}/2 \end{array}\right)$

acts by moving point upstairs

$$\pi_1 BG \longrightarrow [X, BG]_* \longrightarrow [X, BG] \longrightarrow \pi_0 BG$$

$\pi_1 BG \cong \pi_0 G$

$\left(\Omega BG \underset{w.e.}{\simeq} G \quad (\text{HW}) \right)$

<u>Consequence</u> $[X, BG] \equiv$ (unpointed) principle G-bundles $/X$

<u>i.e.</u> topological groups "deloop"

<u>example</u>

$$U(1) \circlearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$$
$$\downarrow$$
$$\mathbb{C}P^{n+1}$$

$\Longrightarrow U(1) \circlearrowright S^\infty$ (contractible)

$$S^\infty = EU(1)$$
$$\downarrow$$
$$\mathbb{C}P^\infty = BU(1)$$
$$K(\mathbb{Z}, 2)$$

Construction of $BO(n)$

$$\mathrm{Frame}_n(\mathbb{R}^k) = \{\text{Space of orthogonal } n\text{-frames in } \mathbb{R}^k\}$$

$$\cap\!\mid$$

$$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^k)$$

$$\mathrm{Frame}_n(\mathbb{R}^k) \circlearrowleft O(n)$$

$$\downarrow \qquad \text{principle } O(n)\text{-bundle}$$

$$\mathrm{Gr}_n(\mathbb{R}^k) = \mathrm{Frame}_n(\mathbb{R}^k)/O(n)$$

cw cx. $\qquad$ space of n-planes in $\mathbb{R}^k$

$$\mathrm{Frame}_n(\mathbb{R}^\infty) = \varinjlim_k \mathrm{Frame}_n(\mathbb{R}^k)$$

contractible

$$\mathrm{Frame}_1(\mathbb{R}^{k-n+1}) \longrightarrow \mathrm{Frame}_n(\mathbb{R}^k) \longrightarrow \mathrm{Frame}_{n-1}(\mathbb{R}^k)$$

$$\mathrm{Frame}_1(\mathbb{R}^{k-n+1}) = S^{k-n}$$

$$S^\infty \longrightarrow \mathrm{Frame}_n(\mathbb{R}^\infty) \longrightarrow \mathrm{Frame}_{n-1}(\mathbb{R}^\infty)$$

$$BO(n) = Gr_n(\mathbb{R}^\infty)$$

Similarly $\quad BU(n) = Gr_n(\mathbb{C}^\infty)$ $\quad$ classifies $\mathbb{C}$ n-plane bundles

$$BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z},2)$$

$$\left\{ \begin{array}{c} \text{principle} \\ U(1)\text{-bundles}/X \end{array} \right\} \cong \left\{ \mathbb{C}x \text{ line bundles}/X \right\} \cong H^2(X;\mathbb{Z})$$

note

$$\pi_1(K(\pi,n)) \longrightarrow [X, K(\pi,n)]_* \longrightarrow [X, K(\pi,n)] \longrightarrow \pi_0 K(\pi,n)$$

$$\begin{array}{ccc} [X, K(\pi,n)]_* & & [X, K(\pi,n)] \\ \| & & \| \\ \tilde{H}^n(X;\pi) & & H^n(X;\pi) \end{array}$$