19 - Chern classes

Note Title 4/13/2010

Recall that a <u>characteristic</u> <u>class</u> ξ of n-plane bundles (real, cx)

is a natural transformation

$$\xi : \mathrm{Vect}_n(-) \longrightarrow H^i(-;R)$$

(i.e.) $\begin{matrix} V \\ \downarrow \\ X \end{matrix} \rightsquigarrow \xi(V) \in H^i(X;R)$

$$X \xrightarrow[f]{} Y \implies \xi(f^*V) = f^*(\xi(V))$$

$$\left.\begin{array}{l} \mathrm{Vect}^{\mathbb{C}}_n(X) = [X, BU(n)] \\ H^i(-;R) = [X, K(R,i)] \end{array}\right\} \xrightarrow{\text{Yoneda}} \xi \in [BU(n), K(R,i)] = H^i(BU(n);R)$$

Recall

$$\begin{tikzcd} V \arrow[r] \arrow[d] & EU(n) \times_{U(n)} \mathbb{C}^n = V_{\text{univ}} \arrow[d] \\ X \arrow[r, "f"'] & BU(n) \end{tikzcd}$$

$$V = f^* V_{\text{univ.}}$$

<u>Moral</u>: $\{\text{characteristic classes}\} \longleftrightarrow \{H^* BU(n)\}$

e.g. $BU(1) = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$

$H^*(BU(1)) = \mathbb{Z}[c_1]$ $\quad |c_1| = 2$

gives $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ line bundle, $c_1(L) \in H^2(X;\mathbb{Z})$

Thm: There exist characteristic classes

$$c_i(V) \in H^{2i}(X;\mathbb{Z}) \qquad \text{for all } n\text{-dim'l cx v.b.'s } V$$

$1 \le i \le n$

Convention: $c_0(V) = 1 \in H^0(X)$

$c_i(V) = 0, \quad i > n$

Satisfying Cartan Formula:

$$\boxed{c_i(V \oplus W) = \sum_{i_1+i_2=i} c_{i_1}(V)\, c_{i_2}(W)}$$

Rmk: Special case of Cartan:

$$c_i(V \oplus \mathbb{C}) = c_i(V)\cdot \overbrace{c_0(\mathbb{C})}^{1} + c_{i-1}(V)\, \overbrace{c_1(\mathbb{C})}^{0}$$

$$= c_i(V)$$

In particular $c_i(\mathbb{C}^n) = 0 \quad \forall\, i > 0$

Producing the classes

Thm $H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \longmapsto \begin{tikzcd} V \arrow[r] \arrow[d] & V_{univ} \arrow[d] \\ X \arrow[r, "f"'] & BU(n) \end{tikzcd} \longmapsto c_n(V) = f^*(c_n)$$

Chern Classes

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(pf of Thm)

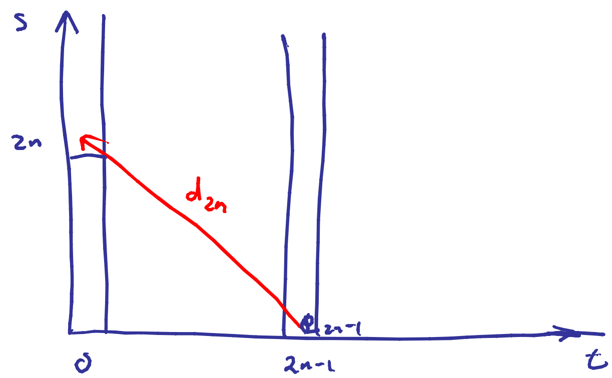
Induction on n ($n=1$ ✓)

$$U(n)/U(n-1) \longrightarrow BU(n-1) \longrightarrow BU(n)$$

$$U(n)/U(n-1) = S^{2n-1}$$

__SSS__

$$H^*(BU(n)) \otimes \Lambda[e_{2n-1}] \Longrightarrow H^* BU(n-1)$$



__deduce:__

$$H^i(BU(n-1)) \xleftarrow{\cong} H^i(BU(n))^{\dagger}$$ ← see edge homo below

for $i < 2n-1$

Define $d_{2n}(e_{2n-1}) = c_n$ ← (Note since $H^*(BU(n-1))$ concentrated in even degrees d_{2n} has to be injective)

$$\Longrightarrow \quad d_{2n}(x\, e_{2n-1}) = c_n x$$

$x \in H^*(BU(n))$

__get:__

$$\mathrm{ann}_{c_n} H^t BU(n) \xrightarrow{\cong} H^{t+2n-1} BU(n-1)$$

$$H^t BU(n-1) \xrightarrow{\cong} H^t BU(n) / c_n H^{t-2n} BU(n)$$

Induction on t shows that

$H^* BU(n-1)$ concentrated in even degrees

$$\Longrightarrow \begin{cases} H^* BU(n) \text{ concentrated in even degrees} \\ \operatorname{ann} c_n = 0 \end{cases}$$

So we have

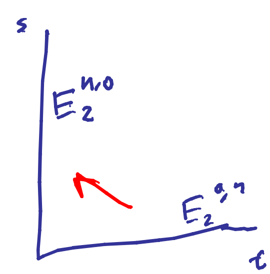
$$0 \to H^{t-2n} BU(n) \xrightarrow{c_n} H^t BU(n) \longrightarrow H^t BU(n-1) \to 0$$

deduce $H^* BU(n) = \mathbb{Z}[c_1, \dots, c_n]$

$\square$

Aside: Edge Homomorphisms $F \xrightarrow{i} E \xrightarrow{p} B$

$$H^s(B; H^t(F)) \Rightarrow H^{s+t}(E)$$



$$\begin{tikzcd} H^n E & E_\infty^{n,0} \arrow[l, hook'] & E_2^{n,0} \arrow[l, two heads] \\ & & H^n(B) \arrow[u, phantom, "\shortparallel"] \arrow[ull, "p^*"] \end{tikzcd}$$

$$\begin{tikzcd} H^n E \arrow[r, two heads] \arrow[rr, bend right, "i^*"'] & E_\infty^{0,n} \arrow[r, hook] & E_2^{0,n} \arrow[d, phantom, "\shortparallel"] \\ & & H^n(F) \end{tikzcd}$$

So we conclude:

$$BU(n-1) \hookrightarrow BU(n)$$

$$\mathbb{Z}[c_1, \dots, c_{n-1}] \longleftarrow \mathbb{Z}[c_1, \dots, c_n]$$

$$c_i \longleftarrow\!\shortmid c_i$$

$$0 \longleftarrow\!\shortmid c_n$$

<u>Lemma</u> Suppose $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ is classified by

$$X \xrightarrow[f]{} BU(n-1)$$

then the composite $X \to BU(n-1) \to BU(n)$

classifies $V \oplus \mathbb{R}$.

<u>pf</u>:

P = principle $U(n-1)$-bundle

$$V = P \times_{U(n-1)} \mathbb{C}^{n-1}$$

$$P \times_{U(n-1)} U(n) \times_{U(n)} \mathbb{C}^n = P \times_{U(n-1)} \mathbb{C}^n$$
$$= \left(P \times_{U(n-1)} \mathbb{C}^{n-1}\right) \times \mathbb{C} \qquad \square$$

Cor:

$$c_i(V \oplus \mathbb{R}) = \begin{cases} c_i(V), & i < n \\ 0, & i = n \end{cases}$$

i.e. chern classes for rank $(n-1)$ bundles compatible w/ chern classes for rank n-bundles

$\Longrightarrow$ can define $c_i(V)$ $\forall\ i \in \mathbb{Z}$

$$\left(c_i(V) = 0 \quad i > rk(V) \right)$$

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proof of Cartan formula is deferred.....
(need more technology)

Splitting Principle

"Any formula involving chern classes need only be checked on sums of line bundles"