20 - Splitting principle + elementary symmetric poly's

Note Title 4/15/2010

Summary:

We computed $H^* BU(n) = \mathbb{Z}[c_1, \ldots, c_n]$

$|c_i| = 2i$

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V^n_{univ} \arrow[d] \\ X \arrow[r, "f"] & BU(n) \end{tikzcd}$$

$$H^{2i}(X) \ni c_i(V) := \begin{cases} f^* c_i & , 1 \le i \le n \\ 0 & , i > n \\ 1 & , i = 0 \end{cases}$$

Satisfying:

Naturality: $c_i(g^* V) = g^* c_i(V)$

dim: $c_i(V) = 0$ if $i > \dim V$

Stability: $c_i(V) = c_i(V \oplus \mathbb{C})$

Normalization:

$$\begin{array}{c} L_{univ} \\ \downarrow \\ \mathbb{C}P^\infty \\ \wr \\ H^* = \mathbb{Z}[x] \end{array} \qquad c_1(L_{univ}) = x$$

$U(n-1) \hookrightarrow U(n)$

$BU(n-1) \longrightarrow BU(n)$

$i < n$ $\quad c_i \longleftarrow c_i$

$0 \longleftarrow c_n$

Goal: Cartan formula:

$$c_n(V \oplus W) = \sum_{n_1 + n_2 = n}' c_{n_1}(V)\, c_{n_2}(W)$$

Thm: (Splitting principle)
V = rk n cx V.B. $/X$

$\exists \quad \tilde{X} \xrightarrow[f]{} X$ s.t.

(1) $f^* V \cong L_1 \oplus \cdots \oplus L_n \qquad \mathrm{rk}(L_i) = 1$

(2) $f^* : H^*(X) \longrightarrow H^*(\tilde{X})$ is injective

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(pf) Suffices to prove you can split off one line.

$$\mathbb{C}P^{n-1} \longrightarrow P(V) \xrightarrow[g]{} X$$

projective space bundle

$$P(V) = \{ (x, L_x) \mid x \in X,\ L_x \subseteq V_x \text{ line} \}$$

$$\begin{array}{l} g^* V \underset{\text{subbundle}}{\supset} \{ (x, L_x, v) \mid v \in L_x \} =: L \\ \downarrow \\ P(V) \end{array}$$

Give $g^* V$ a hermitian structure (can do this see $BU(n)$)

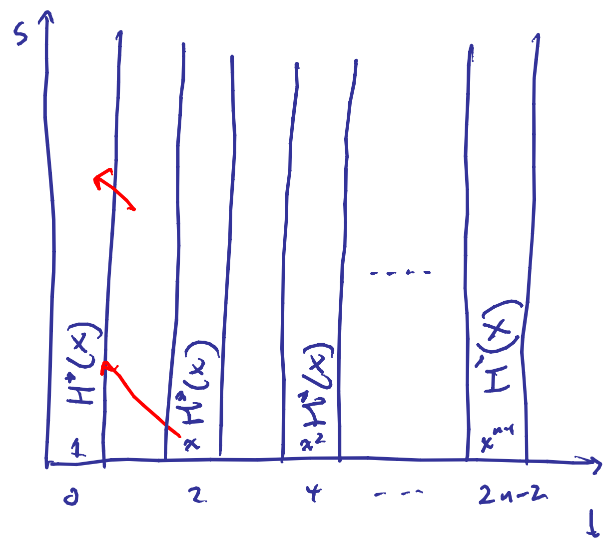
$\Rightarrow \quad g^* V \cong L \oplus L^{\perp}$

Just need to show:

$$g^* : H^*(X) \longrightarrow H^*(P(V))$$

is <u>injective</u>.

<u>SSS</u> $\quad H^s(X; H^t(\mathbb{C}P^{n-1})) \Rightarrow H^{s+t}(P(V))$



$$H^*(\mathbb{C}P^{n-1}) = \frac{\mathbb{Z}[x]}{(x^n)}$$

<u>Edge Homomorphism</u> $\Rightarrow$ need to argue no differentials

(suffices to show x is a P.C.)

Room for d_3 but...

Consider universal example:

$$\mathbb{C}P^{n-1} \longrightarrow P(V_{univ}) \xrightarrow{g_{univ}} BU(n)$$

SSS for g_{univ} concentrated in

$E_2^{s,t}(g_{univ})$ s.t. even

$\Rightarrow$ no diff'ls

$\Rightarrow$ x_{univ} P.C.

$$\begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r, dash] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ P(V) \arrow[r] \arrow[d] & P(V_{univ}) \arrow[d] \\ X \arrow[r] & BU(n) \end{tikzcd} \quad \Rightarrow \quad E_2^{s,t}(g_{univ}) \to E_2^{s,t}(g)$$

$$x_{univ} \longmapsto x$$

$\Rightarrow$ x P.C.

$\square$

Splitting principle $\Rightarrow$ just need to prove

$$c_n(V \oplus W) = \sum_{n_1+n_2=n} c_{n_1}(V)\, c_{n_2}(W)$$

for

V, W = sums of line bundles