3 - Pointed spaces and htpy gps

Note Title 2/9/2010

Last time: Redefined Top, $X \times Y$, $\underline{\mathrm{Map}}(X,Y)$

s.t.

$$\mathrm{Map}(X \times Y, Z) \xrightarrow[\cong]{} \mathrm{Map}(X, \underline{\mathrm{Map}}(Y,Z))$$

Top_* complete cocomplete, ✓

Top_* = ctgy objects = $\{(X,x) \mid x \in X \in \mathrm{Top}\}$

morphisms = bspt preserving

[abusively: $* \in X$ will denote bspt.]

$\underline{\mathrm{Top}_*}$ complete, cocomplete

$X \in \mathrm{Top}_*^{\,I}$

$$\varprojlim^{\mathrm{Top}_*} X = \varprojlim^{\mathrm{Top}} X$$

$\|$

$$\left\{ (x_i)_{i \in I} \;\middle|\; \begin{array}{l} x_i \in X(i) \\ \forall\, \alpha : i \to j \\ \alpha_*(x_i) = x_j \end{array} \right\} \longrightarrow (*)_{i \in I} \in \varprojlim X$$

bspt.

$$\varinjlim^{Top_*} X = \coprod_{i \in I} X(i) \Big/ \sim$$

$\forall\ \alpha : i \to j$

$x_i \in X(i) \qquad x_i \sim \alpha_*(x_i)$

and $\forall\ i, j$

$*_{X(i)} \sim *_{X(j)}$

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e.g. coproduct

$$X \amalg_{Top_*} Y = X \vee Y \qquad \text{"wedge"}$$

$$X \vee Y = X \amalg_{Top} Y \Big/ *_X \sim *_Y$$

Defn

$$\underline{Map}_*(X, Y) \subseteq \underline{Map}(X, Y)$$

$\uparrow$ Top_* $\quad$ subspace of pointed maps

Def smash product $X, Y \in Top_*$

$$X \wedge Y := \frac{X \times Y}{X \times * \cup * \times Y} \in Top_*$$

bspt: collapse point

The following explains why smash product is important

<u>Prop</u>: there is an adjunction

$$\mathrm{Map}_*(X\wedge Y, Z) \cong \mathrm{Map}_*(X, \underline{\mathrm{Map}}_*(Y,Z))$$

(pf)

$$\begin{tikzcd}
X\times * \cup * \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X\times Y \arrow[d] \arrow[ddr, bend left, "f"] & \\
* \arrow[r] \arrow[drr, bend right] & X\wedge Y \arrow[dr, dashed] & \\
 & & Z
\end{tikzcd}$$

$$f \longleftrightarrow \quad \tilde{f} : X \longrightarrow \underline{\mathrm{Map}}(Y,Z)$$

(i) $\left[\begin{array}{l} x \longmapsto (* \longmapsto f(x,*)) \\ \qquad\qquad\qquad\quad \overset{\shortparallel}{*} \\ \Longleftrightarrow \tilde{f}(x) \in \underline{\mathrm{Map}}_*(Y,Z) \end{array}\right.$

(ii) $\left[\begin{array}{l} * \longmapsto (y \longmapsto f(*,y)) \\ \qquad\qquad\qquad\quad \overset{\shortparallel}{*} \\ \Longleftrightarrow \tilde{f} \text{ pointed} \end{array}\right.$

$\square$

There is an adjunction:

$$(-)_+ : \mathrm{Top} \rightleftarrows \mathrm{Top}_* : \text{forget}$$

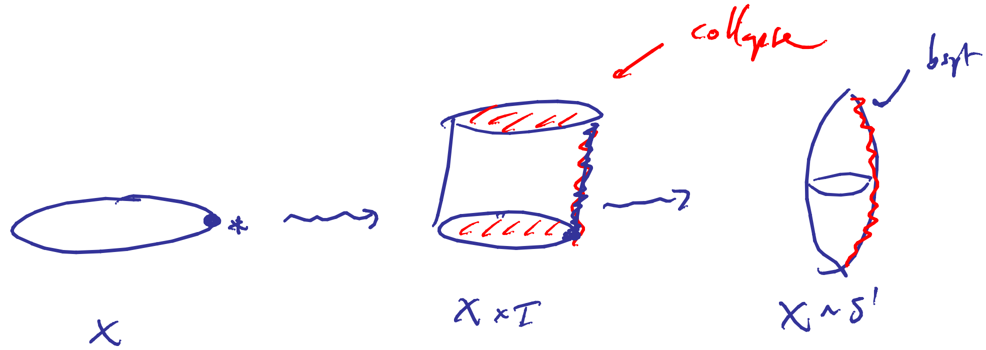
$$X_+ = \text{add disjoint bspt}$$

$$\mathrm{Map}_*(X_+, Y) \cong \mathrm{Map}(X, Y)$$

<u>**Def:**</u> <u>suspension</u>

$$\Sigma : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$$

$$X \longmapsto X \wedge S^1$$



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e.g. $X = S^1$

$\Sigma X \approx S^2$

In general $\Sigma S^n \approx S^{n+1}$

Also

$$X \wedge S^0 \approx S^0$$

$$X_+ \wedge Y_+ \approx (X \times Y)_+$$

$f, g : X \longrightarrow Y$ pointed maps

A pointed htpy $H : X \times I \longrightarrow Y$

satisfies $H_t : X \to Y$ pointed

i.e. $\tilde{H} : I \longrightarrow \underline{\mathrm{Map}}_*(X, Y)$

not a pointed map

$\updownarrow$

$I_+ \longrightarrow \underline{\mathrm{Map}}_*(X, Y)$ pointed

$\updownarrow$

$$\boxed{H : X \wedge I_+ \longrightarrow Y}$$

pointed homotopies are conveniently expressed this way.

$[X, Y]$ = htpy classes of maps $X \to Y$

$[X, Y]_*$ = pointed htpy classes of maps

Def: ΩX = based loop space.

$$\Omega : Top_* \longrightarrow Top_*$$
$$X \longmapsto \underline{Map}_*(S^1, X)$$ "space of loops"

Note: $\pi_1 X = \pi_0 \Omega X$

There is an adjunction

$$\Sigma : Top_* \rightleftarrows Top_* : \Omega$$

$$\begin{array}{ccc} \Sigma X \longrightarrow Y & \longleftrightarrow & X \longrightarrow \Omega Y \\ \parallel & & \parallel \\ X \wedge S^1 & & \underline{Map}_*(S^1, Y) \end{array}$$

Check: $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$

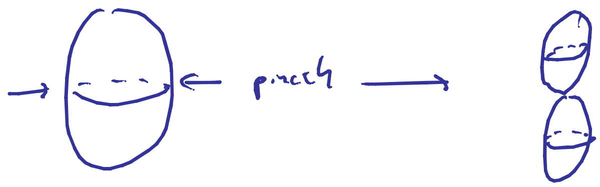
Homotopy grps

$n \geq 1$

$$\pi_n(X) = [S^n, X]_*$$

π_n is a group:

$$S^n \longrightarrow S^n \vee S^n$$



$$\alpha : S^n \longrightarrow X$$

$$\beta : S^n \longrightarrow X$$

$$\alpha + \beta : S^n \xrightarrow[\text{pinch}]{} S^n \vee S^n \xrightarrow[\alpha \vee \beta]{} X$$

$$e : S^n \longrightarrow * \hookrightarrow X$$

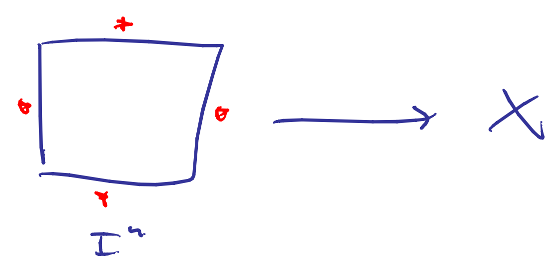
Note: $\pi_0(X) = [S^0, X]_*$

$\pi_0(X)$ is only a pointed set

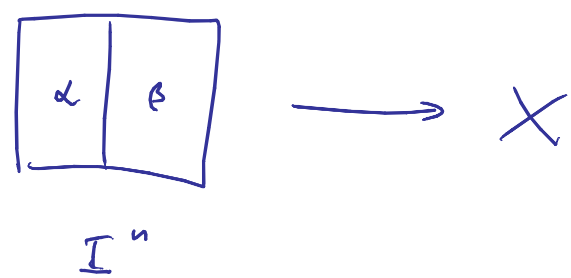
Alternate description

$$\pi_n(X) = [(I^n, \partial I^n), (X, *)]$$

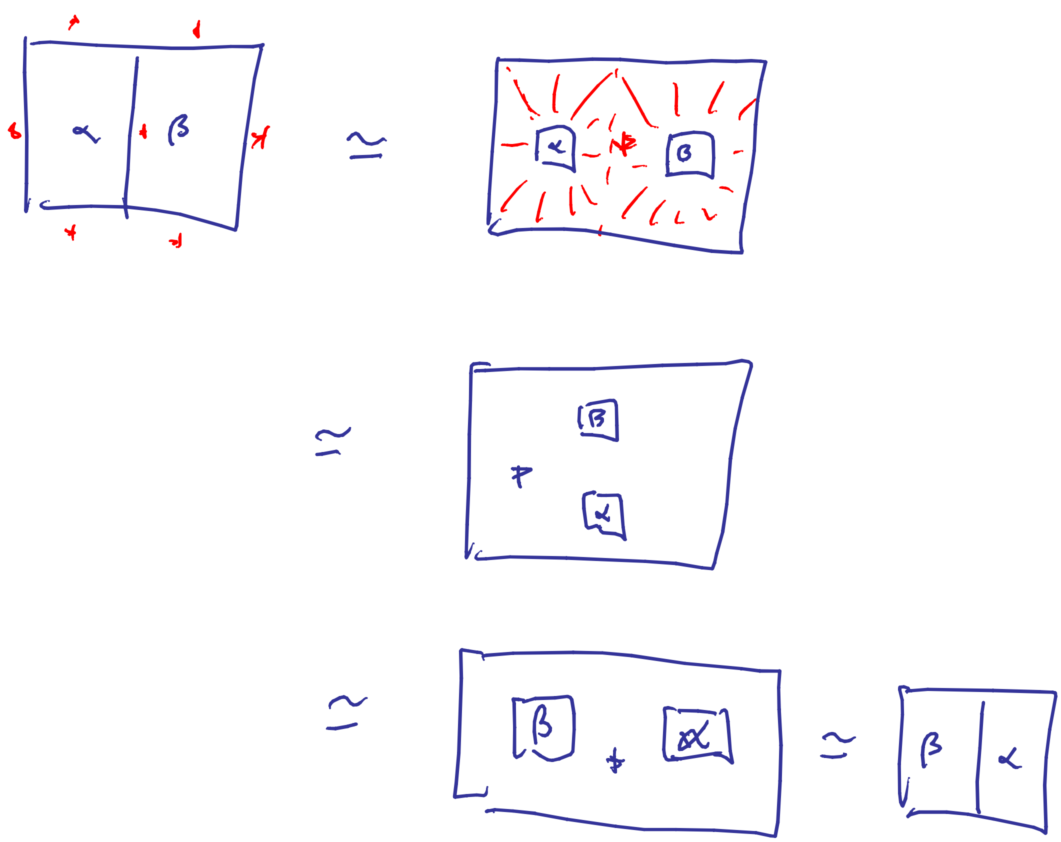
$\alpha \in \pi_n X$



$\alpha + \beta$



prop: $k > 1 \Rightarrow \pi_n X$ abelian



∎

$f \in \mathrm{Map}_*(X, Y)$

$$\Rightarrow \quad f_* : \pi_n X \longrightarrow \pi_n Y$$
$$\alpha \longmapsto f \circ \alpha$$

$\pi_n : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$ is a functor.

<u>Prop</u>: $\pi_{<n}(S^n) = 0$

pf: smooth approximation

<u>Prop</u>: $p : X \longrightarrow Y$ covering space

$\Rightarrow \quad p_* : \pi_k(X) \longrightarrow \pi_k(Y)$

is
- monomorphism $k = 1$
- isomorphism $k \geq 2$

($k = 1$: covering space theory)

Pf: Since S^n is simply connected $(n \geq 2)$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ S^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}", "\exists !"'] & Y \end{tikzcd}$$

$\Rightarrow$ p_* surjective

Furthermore: given a homotopy $H : p_*(\beta) \simeq p_*(\beta')$

HLP lifting property:

$$\begin{tikzcd} S^n \wedge \{0\}_+ \arrow[r] \arrow[d] & X \arrow[d] \\ S^n \wedge I_+ \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}", "\exists !"'] & Y \end{tikzcd}$$

simply connected

uniqueness of lifts $\Rightarrow$ $\tilde{H}_1 = \beta'$

$\Rightarrow$ p_* injective

$\square$

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ex: $\mathbb{R} \longrightarrow S^1$ covering

$$\Rightarrow \quad \pi_k(S^1) = \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$

More generally, any space w/ a contractible universal cover has: $\pi_{>1} X = 0$

e.g. $\pi_k(T^2) = \begin{cases} \mathbb{Z}\times\mathbb{Z}, & k=1 \\ 0, & o/w \end{cases}$

Alternatively use $T^2 = S^1 \times S^1$ and:

Prop

$$\pi_k(X_1 \times X_2) = \pi_k(X_1) \times \pi_k(X_2)$$

(pf) universal property of $- \times - \Rightarrow$

$$\{\alpha : S^n \to X_1 \times X_2\} \longleftrightarrow \{\alpha_i : S^n \to X_i\}$$

$$\{H : S^n \wedge I_+ \to X_1 \times X_2\} \longleftrightarrow \{H_i : S^n \wedge I_+ \to X_i\}$$

$\square$

In general

$$\pi_4(S^4) = \mathbb{Z} \qquad \text{(we will}_{\wedge}^{\text{Eventually}}\text{ see this)}$$

But $\pi_{>4}(S^4)$ is a mess.

(Table: p 339 of Hatcher)

π_* typically more difficult ^or sometimes impossible to compute than H_*

but a more powerful invariant

(we will prove π_*-iso $\Longrightarrow$ H_*-iso)

Easy exercise (in ANY category)

$$X \in \mathcal{C}^I$$

$$\mathrm{Map}_{\mathcal{C}}(\varinjlim X, Z) \cong \varprojlim \mathrm{Map}_{\mathcal{C}}(X, Z)$$

$$\mathrm{Map}_{\mathcal{C}}(Z, \varprojlim X) \cong \varprojlim \mathrm{Map}_{\mathcal{C}}(Z, X)$$

However $\varinjlim \mathrm{Map}_{\mathcal{C}}(Z, X_i) \longrightarrow \mathrm{Map}_{\mathcal{C}}(Z, \varinjlim X)$

is typically not an isomorphism

Lem Let $X_0 \to X_1 \to \cdots$

be a sequence of closed inclusions in Top.

Suppose K is cpt

$$\Rightarrow \quad \varinjlim \mathrm{Map}(K, X_i) \longrightarrow \mathrm{Map}(K, \underset{\substack{\| \\ \bigcup X_i}}{\varinjlim} X_i)$$

is an isomorphism.

Clearly an injection.

closed inclusions $\Rightarrow$ weak top. (automatically w. Haus)

just need to check

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, dashed] & \bigcup_i X_i \\ & X_j \arrow[u] \end{tikzcd} \qquad X, \quad \exists j$$

Now:

Suppose not. i_k increasing

$\Longrightarrow$ we can pick $x_{i_k} \in f(K) \cap (X_{i_k} - X_{i_k - 1})$

Consider $C = \{x_{i_1}, x_{i_2}, \ldots\}$

$C \cap X_{i_k} = \{x_{i_1}, \ldots, x_{i_k}\}$ closed

$\Longrightarrow$ C closed

Consider $U_k = X - \{x_{i_{k+1}}, x_{i_{k+2}}, \ldots\}$

open

U_k covers $f(K)$, no finite subcover does.

$\to\leftarrow$ $\square$

Cor: $X_0 \to X_1 \to \cdots$ seq. closed inclusions

$$\varinjlim \pi_k(X_i) = \pi_k(\varinjlim X_i)$$

(pf) S^k, $S^k \times I$ cpt.

$\square$

Cor: $S^\infty = \varinjlim (S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots)$

$\pi_k(S^\infty) = 0$

(pf) $\pi_k(\varinjlim_n S^n) = \varinjlim_n (\pi_k S^n)$

zero for $n > k$ $\square$

Cor: $\pi_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2, & k = 1 \\ 0, & k \neq 1 \end{cases}$

(pf) $S^\infty \longrightarrow \mathbb{R}P^\infty$ universal 2-fold cover

$(S^n \longrightarrow \mathbb{R}P^n)$

$\Rightarrow$ $\pi_1 \mathbb{R}P^\infty = \mathbb{Z}/2$

$\pi_k \mathbb{R}P^\infty = \pi_k S^\infty = 0$ $\quad k \geq 2$

Rmk: $X \in \mathrm{Top}_*$ satisfies

$\pi_k(X) = \begin{cases} \pi, & k = n \\ 0, & k \neq 1 \end{cases}$

is called Eilenberg-Maclane "$K(\pi, n)$". We shall see these are unique up to homotopy. (always exist)