4 - fundamental groupoid - htpy equivalence | Relative π_*

Action of fundamental gpoid

Recall G = gp $\rightsquigarrow$ $\underline{G}$ cat

one object $*$

$\mathrm{Mor}(*,*) = G$

$\{\text{gps}\} = \{\text{categories w/ one object, all morphisms are isomorphisms}\}$

$\cap| \qquad\qquad \cap|$

$\{\text{groupoids}\} = \{\text{categories where all morph are isos}\}$

Def: $X \in \mathrm{Top}$: the fundamental groupoid $\pi_{\le 1} X$

is defined to be the groupoid

objects = points of X

$$\mathrm{Map}_{\pi_{\le 1} X}(x,y) = \left\{ \begin{array}{l} \text{paths } \gamma : I \to X \\ \gamma(0) = x \\ \gamma(1) = y \end{array} \right\} \Big/ \text{htpy rel } \partial I$$

Composition = path composition

$$\begin{tikzcd} x \arrow[r, "{[\gamma]}"'] \arrow[rr, bend left, "{[\gamma'\gamma]}"] & y \arrow[r, "{[\gamma']}"'] & z \end{tikzcd}$$

identity morph = const paths

Note: $\pi_{oid}(x,x) \cong \pi_1(X)$

$$\begin{array}{c}\pi_{oid}(x,x) \\ \Psi \\ [\gamma : x \longmapsto x]\end{array}$$

Rmk: depends on convention!

for me $\gamma_1, \gamma_2 \in \pi_1$

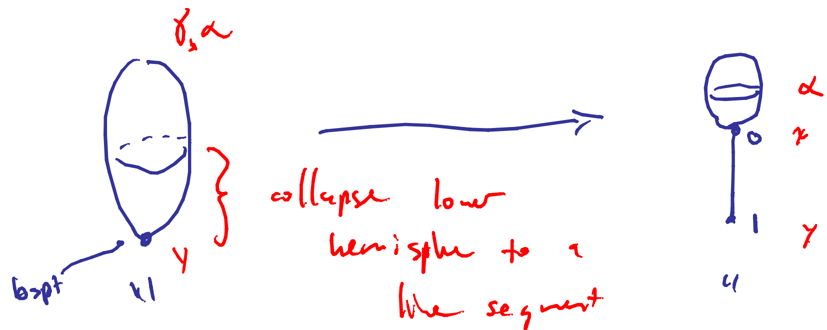
$\gamma_1 \gamma_2$ means

go around γ_2, then γ_1

$\gamma : x \longrightarrow y$ in X

induces: $\gamma_* : \pi_k(X,x) \longrightarrow \pi_k(X,y)$

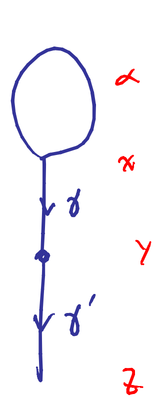
Defn



$S^n \qquad\qquad S^n \vee I$

$\alpha : (S^n, *) \longrightarrow (X, x)$

set $\gamma_*\alpha : S^n \longrightarrow S^n \vee I \xrightarrow[\alpha \cup \gamma]{} X$



$$(\gamma'\gamma)_*[\alpha] = \gamma'_* \, \gamma_*[\alpha]$$

$\Longrightarrow$ $X \in \mathrm{Top}$

π_k gives a functor

$$\pi_{oid}(X) \longrightarrow Gp$$

$$x \longmapsto \pi_k(X, x)$$

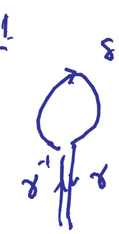
In particular,

$\pi_1(X, x)$ acts on $\pi_k(X, x)$

$k \geq 1$

Note

k = 1



$\delta \in \pi_1(X, x)$

$\gamma \in \pi_1(X, x)$

$$\Rightarrow \gamma_*[\delta] = \gamma \delta \gamma^{-1}$$

action is by conjugation.

Also: since π_{oid} is a groupoid, $\gamma : x \to y$

$$\gamma_* : \pi_k(X, x) \xrightarrow[\cong]{} \pi_k(X, y)$$

is an iso.

$\Rightarrow$ iso class of $\pi_k(X, x)$ only depends on path comp. of x.

Weak equivalences

Def

$X, Y \in \mathrm{Top}$. A map $f : X \to Y$ is said to be a weak (htpy) equivalence if

(1) $f_* : \pi_0 X \longrightarrow \pi_0 Y$ is a bijection

(2) $\forall \ x \in X$

$$f_* : \pi_k(X, x) \longrightarrow \pi_k(Y, f(x)) \text{ is iso.}$$

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prop: $\left(f : X \to Y \text{ htpy equiv.} \right) \Longrightarrow \left(f \text{ is a w.e.} \right)$

(pf) Let $g : Y \longrightarrow X$ be htpy inverse

$$H : gf \simeq 1_X$$

$$H' : fg \simeq 1_Y$$

pick $x \in X$

$$\pi_k(X, x) \xrightarrow[f]{} \pi_k(Y, f(x)) \xrightarrow[g]{} \pi_k(X, gf(x))$$

htpy H gives a commutative diagram

$$\begin{tikzcd} \pi_k(X,x) \arrow[r, "f_*"] \arrow[drr, "\cong", "(1_X)_*"'] & \pi_k(Y, f(x)) \arrow[r, "g_*"] & \pi_k(X, gf(x)) \arrow[d, "\cong"', "\gamma_*"] \\ & & \pi_k(X,x) \end{tikzcd}$$

$$\gamma = H|_{\{x\}} : I \longrightarrow X$$

$\Longrightarrow$ $g_* f_*$ isomorphism $\Rightarrow$ f_* monic

similarly show $f_* g_*$ iso $\Longrightarrow$ f_* epic

$\square$

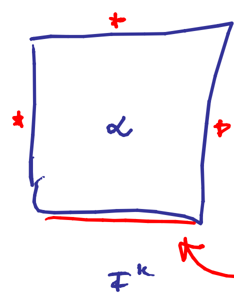
Relative Htpy gps:

$$A \hookrightarrow X \quad \text{in } \mathrm{Top}_*$$

$$\pi_k(X,A) = \left[\left(I^k, \partial I^k, \partial I^k - I^{k-1}\times\{0\}\right), (X, A, *) \right]$$

e.g. $[\alpha] \in \pi_k(X,A)$

$\alpha \sim$



one face only maps into A

rest of faces map to $*$

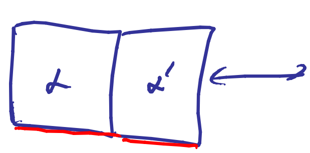
only works for $k > 0$!

$k = 1$: this is a pointed set

$k \geq 2$: this is a gp

$k \geq 3$: abelian

addition



$\alpha' + \alpha$

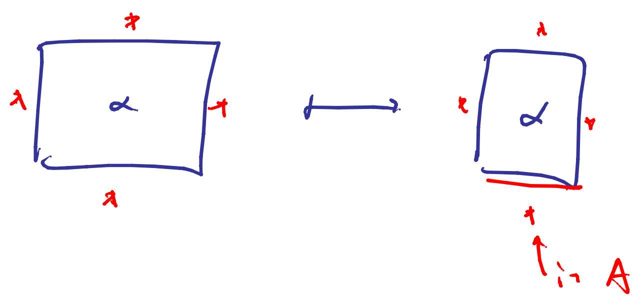
LES of a pair:

$$i : A \hookrightarrow X$$

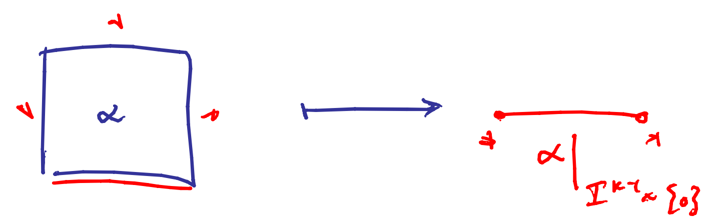
Then ∃ a LES:

$$\cdots \xrightarrow[\partial]{} \pi_k(A) \xrightarrow[i_*]{} \pi_k(X) \xrightarrow[j]{} \pi_k(X,A) \xrightarrow[\partial]{} \pi_{k-1}(A) \xrightarrow[i_*]{} \cdots$$

j :



∂ :



Sequence ends w/

$$\pi_1(A) \xrightarrow[i_*]{} \pi_1(X) \xrightarrow[\partial]{} \pi_1(X,A) \xrightarrow[j]{} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

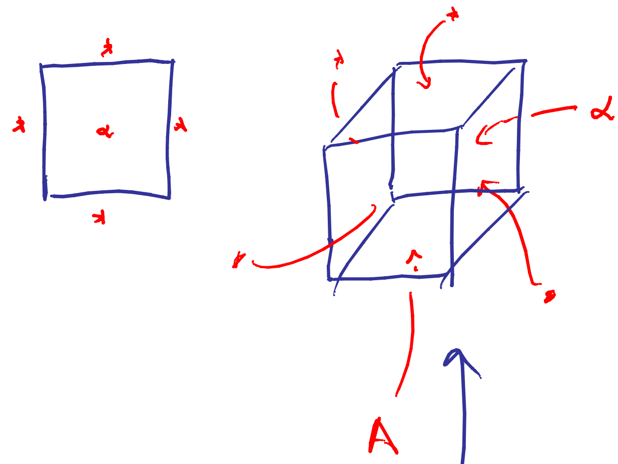
pointed sets

ker = im

w/ "ker" = preimage of bspt

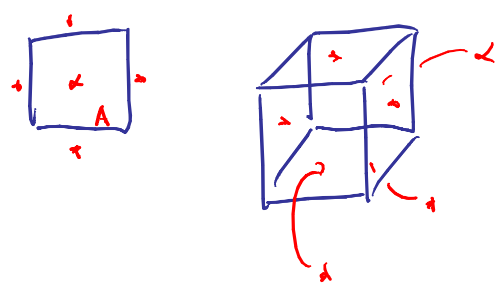
(Sketch)

$$\pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X,A)$$



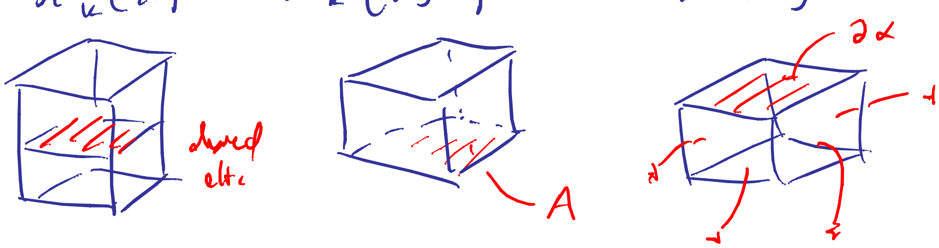
gives htpy between α and elt of $\pi_k(A)$

$$\pi_k(X,A) \longrightarrow \pi_{k-1}(A) \longrightarrow \pi_{k-1}(X)$$



this is an elt of $\pi_k(X,A)$

$$\pi_k(X) \longrightarrow \pi_k(X,A) \longrightarrow \pi_{k-1}(A)$$



Whitehead product (won't use these)

$S^k \times S^n$ fin CW cx

0-cell

k-cell

n-cell

k+n-cell

$$\begin{tikzcd} S^{k+n-1} \arrow[r, "a_{k,n}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S^k \vee S^n \arrow[d] \\ D^{k+n} \arrow[r] & S^k \times S^n \end{tikzcd}$$

$\alpha \in \pi_k(X)$

$\beta \in \pi_n(X)$, $[\alpha, \beta] \in \pi_{k+n-1}(X)$

$$[\alpha,\beta] : S^{k+n-1} \longrightarrow S^k \vee S^n \xrightarrow[\alpha \vee \beta]{} X$$

"$\pi_*(X)$ is a graded Lie algebra"