2. Categories

"Homotopy category" [May: Chapter 2]

"Representable functor"

Need to discuss notion of ctgy.

Def A category $\mathcal{C}$ consists of

(1) a collection $\mathrm{Ob}\,\mathcal{C}$

(2) for any pair $x, y \in \mathrm{Ob}\,\mathcal{C}$
a set of morphisms $\mathrm{Map}_{\mathcal{C}}(x,y)$

(3) for each $x \in \mathrm{Ob}\,\mathcal{C}$, an identity morphism
$1_x \in \mathrm{Map}_{\mathcal{C}}(x,x)$

(4) $x, y, z \in \mathrm{Ob}\,\mathcal{C}$, a composition map:

$$\mathrm{Map}_{\mathcal{C}}(y,z) \times \mathrm{Map}_{\mathcal{C}}(x,y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(x,z)$$
$$(f,g) \longmapsto f \circ g$$

Satisfying:

- (Identity) $f \in \mathrm{Map}_{\mathcal{C}}(x,y)$

$$1_y \circ f = f \circ 1_x = f$$

- (associativity) for compatible f, g, h, $(f \circ g) \circ h = f \circ (g \circ h)$

Examples

Sets : ob = sets

morph = set maps

Gp : ob = gps

morph = gp homomorphisms

Ab : ob = ab gps

morph = homomorphisms

Mod_R : ob = R-modules (R = ring)

morph = R-mod homos

Top : ob = top'l spaces

morph = cts maps

If G is a gp, can form a cat

$\underline{G}$ $\quad$ ob $\underline{G} = *$

$$\text{Map}_{\underline{G}}(*, *) = G$$

If $\mathcal{C}$ is a cat, $f : x \to y$ is an <u>iso</u>

if $\exists$ morphism $f^{-1} : y \to x$, s.t. $ff^{-1} = 1_y$

$f^{-1}f = 1_x$

"Ctgy of categories?"

Morphisms between cats: Functors

<u>Def:</u> Let $\mathcal{C}, \mathcal{D}$ be cats

a (covariant) (contravariant) functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is a map

$$F : \mathrm{ob}\,\mathcal{C} \longrightarrow \mathrm{ob}\,\mathcal{D}$$

together with set maps

$$F : \mathrm{Map}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(F(x), F(y))$$

$\mathrm{Map}_{\mathcal{D}}(F(y), F(x))$

satisfying

- $F(1_x) = 1_{F(x)}$
- $F(f \circ g) = F(f) \circ F(g)$

$F(g) \circ F(f)$

$\mathcal{C}$ = cat

$\mathcal{C}^{op}$ = opposite cat:

$$\mathrm{ob}\,\mathcal{C}^{op} = \mathrm{ob}\,\mathcal{C}$$

$$\mathrm{Map}_{\mathcal{C}^{op}}(x, y) = \mathrm{Map}_{\mathcal{C}}(y, x)$$

There is a bijective correspondence

$$\{\text{contravariant functors } \mathcal{C} \to \mathcal{D}\}$$
$$\updownarrow$$
$$\{\text{covariant functors } \mathcal{C}^{op} \to \mathcal{D}\}$$

Examples

$\mathcal{U} : \text{Gps} \longrightarrow \text{Sets}$ forgetful functor

$\mathcal{U}(G) =$ underlying set of G

$F : \text{Sets} \longrightarrow \text{Gps}$ free gp functor

$F(S) =$ free gp generated by S

$A \in \text{Ab}$

$$\text{Hom}(A, -) : \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(A, B)$$

covariant functor (why?)

$$\text{Hom}(-, A) : \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(B, A)$$

contravariant functor

$$\text{Hom}(-,-) : \text{Ab}^{op} \times \text{Ab} \longrightarrow \text{Ab}$$

product cat.

$$\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$$
$$\text{Map}_{\mathcal{C}\times\mathcal{D}}((x, x'), (y, y')) = \text{Map}_{\mathcal{C}}(x, y) \times \text{Map}_{\mathcal{D}}(x', y')$$

$\pi_1 : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$

ctgy of pointed spaces and bspt preserving maps

$(X, x_0) \longmapsto \pi_1(X, x_0)$

$H^k(-;R) : \mathrm{Top} \longrightarrow \mathrm{Mod}_R$

$X \longmapsto H^k(X;R)$

Functors are maps between categories

Natural transformations are maps between functors

Def $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ (covariant) functors

A natural transformation

is a collection of morphisms in $\mathcal{D}$

$\eta_x : F(x) \longrightarrow G(x)$

$\forall\ x \in \mathrm{Ob}\,\mathcal{C}$

that satisfy: $\forall$ morphism $f : x \to y$ in $\mathcal{C}$

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

e.g. Suppose $A, A' \in Ab$

$\alpha : A \longrightarrow A'$ homomorphism

get natural transformation

$$\alpha_* : Hom(-, A) \longrightarrow Hom(-, A')$$

given B

$$(\alpha_*)_B : Hom(B, A) \longrightarrow Hom(B, A')$$
$$f \longmapsto \alpha \circ f$$

HW: "adjoint functors"

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

e.g. $F : \mathrm{Sets} \rightleftarrows \mathrm{Gp} : \mathcal{U}$

$$\mathrm{Map}_{\mathrm{Gp}}(F(S), G) \cong \mathrm{Map}_{\mathrm{Sets}}(S, \mathcal{U}G)$$

$R\times - : \mathrm{Sets} \rightleftarrows \mathrm{Sets} : \mathrm{Map}(R, -)$

$$\mathrm{Map}_{\mathrm{Set}}(R\times S, T) \cong \mathrm{Map}_{\mathrm{Set}}(S, \mathrm{Map}(R,T))$$

$$f \longmapsto (s \longmapsto (r \longmapsto f(r,s)))$$

$A\otimes - : \mathrm{Ab} \rightleftarrows \mathrm{Ab} : \mathrm{Hom}(A, -)$

$$\mathrm{Hom}(A\otimes B, C) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$$

"tensor product is dual to hom"

importance of $\otimes$

Products, coproducts, limits, colimits

$\mathcal{C}$ = cat, I = small category

$\mathcal{C}^I$ = category $\begin{cases} \text{objects} = \text{Functors } I \to \mathcal{C} \\ \text{Morphs} = \text{natural transformations} \end{cases}$

category of "I-shaped diagrams in $\mathcal{C}$"

e.g. $I = \underline{G}$

Discussion: what is $\mathrm{Set}^{\underline{G}}$

i.e. $\mathrm{Map}(i,j) = \begin{cases} \mathrm{id}, & i=j \\ \emptyset, & \text{o/w} \end{cases}$

$$I = \{0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots\cdots\}$$

$\mathcal{C}^I$ objects: $C(0) \leftarrow C(1) \leftarrow C(2) \leftarrow \cdots\cdots$

Limits

Def: Let $X : I \to \mathcal{C}$ be an I-shaped diagram

$\varprojlim X = \varprojlim_{i \in I} X(i)$ (if it exists) satisfies the following universal property

There exist maps $\varprojlim X \xrightarrow{\pi_i} X(i)$

s.t. $\forall\ \alpha : i \to j$

$$\begin{tikzcd} & X(i) \arrow[d] \\ \varprojlim X \arrow[ur] \arrow[r] & X(j) \end{tikzcd}$$

Given $Y \in \mathcal{C}$, $f_i : Y \to X(i)$

s.t. $\forall\ \alpha : i \to j$ in I

$$\begin{tikzcd} & X(i) \arrow[d, "X(\alpha)"] \\ Y \arrow[ur, "f_i"] \arrow[r, "f_j"'] & X(j) \end{tikzcd}$$

commutes

$\exists! \quad f : Y \to \varprojlim X$

s.t.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "f_i"'] & \varprojlim X \arrow[d, "\pi_i"] \\ & X(i) \end{tikzcd}$$

(commutes) $\forall i$

---

examples of common I's

$\mathrm{Ob}\, I = \{1 \quad 2\}$ no non-identity morphisms

$$\mathrm{Ob}\, \mathcal{C}^I = \{(X_1, X_2) \mid X_1, X_2 \in \mathcal{C}\}$$

Universal property:

$$\begin{tikzcd} & & X_1 \\ Y \arrow[urr, "f_1"] \arrow[rr, dashed, "\exists!"'] \arrow[drr, "f_2"'] & & \varprojlim (X_1, X_2) \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ & & X_2 \end{tikzcd}$$

e.g. $\mathcal{C} = \mathrm{Sets}$

Q: what is $\varprojlim (X_1, X_2)$?

In general we denote

$$\varprojlim (X_1, X_2) := X_1 \times X_2$$

called the (categorical) product in $\mathcal{C}$

---

e.g. $I = G$, $\mathcal{C} = \mathrm{Sets}$

$X \in \mathrm{Sets}^G$

$X \circlearrowleft G$

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, dashed, "\exists !"'] & X \\
 & \varprojlim X \arrow[u, "\pi"']
\end{tikzcd}
\qquad gf = f \quad \forall g$$

What is $\varprojlim X$?

Another important example

$$I = \left(0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots\cdots \right)$$

$X \in \mathcal{C}^I$ $\qquad \varprojlim X$ is called the projective limit.

$$\begin{tikzcd}
 & & & Y \arrow[dlll, "f_0"'] \arrow[dll, "f_1"] \arrow[dl, "f_2"] \arrow[dr, dashed, "\exists!"] & \\
X(0) & X(1) \arrow[l, "\alpha_1"] & X(2) \arrow[l, "\alpha_2"] & \cdots \arrow[l] & \varprojlim X \arrow[l]
\end{tikzcd}$$

e.g. $\quad \mathcal{C} = \text{Sets}$

$$\varprojlim X = \left\{ (x_i)_{i=0}^{\infty} \;\middle|\; \alpha_i(x_i) = x_{i-1} \right\}$$

e.g. $\quad \varprojlim \left(\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \cdots \right)$

$$\parallel$$

$$\hat{\mathbb{Z}}_p$$

Yet another common example

pullback

$$I = \left\{ \begin{tikzcd} & 2 \arrow[d] \\ 1 \arrow[r] & 3 \end{tikzcd} \right\}$$

$$\mathcal{C}^I = \left\{ \begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right\}$$

$$\varprojlim \left(\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right) \quad \text{is denoted} \quad X \times_Z Y$$

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$

e.g. in Top

$$X \times_Z Y = \{(x,y) \in X \times Y \mid \alpha(x) = \beta(y)\} \underset{\text{subspace}}{\subset} X \times Y$$

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & E \arrow[d] \\ * \arrow[r, "b"'] & B \end{tikzcd}$$

denotes $F = E \times_B *$

Colimits

Same but dual