

## 2. Categories

Note Title

2/1/2010

"Homotopy category" [May: Chapter 2]  
"Representable functor"

Need to discuss notion of ctgy.

Def A category  $\mathcal{C}$  consists of

(1) a collection  $\text{Ob } \mathcal{C}$

(2) for any pair  $x, y \in \text{Ob } \mathcal{C}$

a set of morphisms  $\text{Map}_{\mathcal{C}}(x, y)$

(3) for each  $x \in \text{Ob } \mathcal{C}$ , an identity morphism

$1_x \in \text{Map}_{\mathcal{C}}(x, x)$

(4)  $x, y, z \in \text{Ob } \mathcal{C}$ , a composition map:

$\text{Map}_{\mathcal{C}}(y, z) \times \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{C}}(x, z)$

$(f, g) \longmapsto f \circ g$

Satisfying:

• (Identity)  $f \in \text{Map}_{\mathcal{C}}(x, y)$

$$1_y \circ f = f \circ 1_x = f$$

• (associativity)

for composable  $f, g, h$ ,  $(f \circ g) \circ h$

$f \circ (g \circ h)$

## Examples

Sets :  $Ob = \text{sets}$   
morph = set maps

Gp :  $Ob = \text{gps}$   
morph = gp homomorphisms

Ab :  $Ob = \text{ab gps}$   
morph = homomorphisms

Mod<sub>R</sub> :  $Ob = R\text{-modules}$  ( $R = \text{ring}$ )  
morph =  $R$ -mod maps

Top :  $Ob = \text{top'l spaces}$   
morph = cts maps

If  $\mathcal{C}$  is a sp, can form a cat

$$\underline{\mathcal{C}} \quad Ob \underline{\mathcal{C}} = *$$

$$Map_{\underline{\mathcal{C}}}(*, *) = \mathcal{C}$$

If  $\mathcal{C}$  is a cat,  $f: x \rightarrow y$  is an iso

if  $\exists$  morphism  $f^{-1}: y \rightarrow x$ , s.t.  $ff^{-1} = 1_y$   
 $f^{-1}f = 1_x$

"city of categories?"

Morphisms between cuts: Functors

Def: Let  $\mathcal{C}, \mathcal{D}$  be cuts

a <sup>(contravariant)</sup> (covariant) functor

$$F: \mathcal{C} \rightarrow \mathcal{D}$$

is a map

$$F: \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{D}$$

together with set maps

$$F: \text{Map}_{\mathcal{C}}(x, y) \rightarrow \text{Map}_{\mathcal{D}}(F(x), F(y))$$

Satisfying  $\text{Map}_{\mathcal{D}}(F(y), F(x))$

$$\bullet F(1_x) = 1_{F(x)}$$

$$\bullet F(f \circ g) = F(f) \circ F(g)$$

$$F(g) \circ F(f)$$

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$\mathcal{C} = \text{cut}$

$\mathcal{C}^{\text{op}} = \text{opposite cut}$

$$\text{ob } \mathcal{C}^{\text{op}} = \text{ob } \mathcal{C}$$

$$\text{Map}_{\mathcal{C}^{\text{op}}}(y, x) = \text{Map}_{\mathcal{C}}(x, y)$$

There is a bijective correspondence

$$\{\text{contravariant functors } \mathcal{C} \rightarrow \mathcal{D}\}$$



$$\{\text{covariant functors } \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}\}$$

### Examples

$$U: \text{Grps} \longrightarrow \text{Sets} \quad \text{forgetful functor}$$

$U(G) = \text{underlying set of } G$

$$F: \text{Sets} \longrightarrow \text{Grps} \quad \text{free gp functor}$$

$F(S) = \text{free gp generated by } S$

$$A \in \text{Ab}$$

$$\text{Hom}(A, -): \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(A, B)$$

Covariant functor (why?)

$$\text{Hom}(-, A): \text{Ab} \longrightarrow \text{Ab}$$
$$B \longmapsto \text{Hom}(B, A)$$

contravariant functor

$$\text{Hom}(-, -): \text{Ab}^{\text{op}} \times \text{Ab} \longrightarrow \text{Ab}$$

$\uparrow$  product cat.

$$\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$$

$$\text{Map}_{\mathcal{C} \times \mathcal{D}}((x, x'), (y, y')) = \text{Map}_{\mathcal{C}}(x, y) \times \text{Map}_{\mathcal{D}}(x', y')$$

$$\mathcal{T}op_* \longrightarrow \mathcal{G}p$$

category of pointed spaces  
and basepoint preserving maps

$$(X, x) \longmapsto \tau_0(X, x)$$

$$H^k(-; \mathbb{R}) : \mathcal{T}op \longrightarrow \mathcal{M}od_{\mathbb{R}}$$

$$X \longmapsto H^k(X; \mathbb{R})$$

Functors are maps between categories

Natural transformations are maps between functors

Def  $F, G : \mathcal{C} \longrightarrow \mathcal{D}$  (Covariant) functors

A natural transformation

is a collection of morphisms in  $\mathcal{D}$

$$\eta_x : F(x) \longrightarrow G(x)$$

$$\forall x \in \text{Ob } \mathcal{C}$$

that satisfy:  $\forall$  morphism  $f : x \rightarrow y$  in  $\mathcal{C}$

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \eta_x \downarrow & & \downarrow \eta_y \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

e.g. Suppose  $A, A' \in \mathcal{A}$

$$\alpha: A \rightarrow A' \quad \text{isomorphism}$$

get natural transformation

$$\alpha_*: \text{Hom}(-, A) \rightarrow \text{Hom}(-, A')$$

given  $B$

$$(\alpha_*)_B: \text{Hom}(B, A) \longrightarrow \text{Hom}(B, A')$$

$$f \longmapsto \alpha \circ f$$

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HW: "adjoint functors"

$$F: \mathcal{C} \rightleftarrows \mathcal{D}: G$$

e.g.

$$F: \text{Sets} \rightleftarrows \text{Grp} : U$$

$$\text{Map}_{\text{Grp}}(F(S), G) \cong \text{Map}_{\text{sets}}(S, UG)$$

$$R_x - : \text{Sets} \rightleftarrows \text{Sets} : \text{Map}(R, -)$$

$$\text{Map}_{\text{set}}(R \times S, T) \cong \text{Map}_{\text{set}}(S, \text{Map}(R, T))$$

$$f \longmapsto (s \longmapsto (r \longmapsto f(r, s)))$$

$$A \otimes - : \text{Ab} \rightleftarrows \text{Ab} : \text{Hom}(A, -)$$

$$\text{Hom}(A \otimes B, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

"tensor product is dual to hom"  
importance of  $\otimes$

Products, coproducts, limits, colimits

$\mathcal{C} = \text{cat}$  ,  $I = \text{small category}$

$$\mathcal{C}^I = \text{caty} \begin{cases} \text{objects} = \text{Functs } I \rightarrow \mathcal{C} \\ \text{Morphs} = \text{natural transformations} \end{cases}$$

category of " $I$ -shaped diagrams  
in  $\mathcal{C}$ "

eg,  $I = \underline{G}$

Discussion: What is

Set  $I$

ie,  $\text{Map}(i,j) = \begin{cases} \neq, & i \leq j \\ \emptyset, & \text{otherwise} \end{cases}$

$$I = \{ 0 \leftarrow 1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \}$$

$\in I$  objects,  $c(0) \leftarrow c(1) \leftarrow c(2) \leftarrow \dots$

Limits

Def:

Let  $X: I \rightarrow \mathcal{C}$  be an  $I$ -shaped diagram

(if it exists)

$$\varprojlim X = \varprojlim_{i \in I} X(i)$$

satisfies the following universal property

There exist maps  $\varprojlim X \xrightarrow{\pi_i} X(i)$

s.t.  $\forall \alpha: i \rightarrow j$

$$\begin{array}{ccc} \varprojlim X & \xrightarrow{\pi_i} & X(i) \\ & \searrow & \downarrow \alpha \\ & & X(j) \end{array}$$

Given  $Y \in \mathcal{C}$ ,  $f_i: Y \rightarrow X(i)$

s.t.

$\forall \alpha: i \rightarrow j$   
in  $I$

$$\begin{array}{ccc} Y & \xrightarrow{f_i} & X(i) \\ & \searrow f_j & \downarrow \alpha \\ & & X(j) \end{array}$$

commutes



$$\exists! \quad f : Y \rightarrow \varprojlim X$$

$$\text{S.t.} \quad \begin{array}{ccc} Y & \longrightarrow & \varprojlim X \\ & \searrow f_i & \downarrow \pi_i \\ & & X(i) \end{array} \quad \begin{array}{l} \text{(commute)} \\ \forall i \end{array}$$

Examples of common  $I$ 's

$$\text{Ob } I = \{1, 2\} \quad \begin{array}{l} \text{no non-identity} \\ \text{morphisms} \end{array}$$

$$\text{Ob } \mathcal{C}^I = \left\{ (X_1, X_2) \mid X_1, X_2 \in \mathcal{C} \right\}$$

Universal property:

$$\begin{array}{ccc} & & X_1 \\ & \nearrow f_1 & \uparrow \pi_1 \\ Y & \dashrightarrow \exists! & \varprojlim (X_1, X_2) \\ & \searrow f_2 & \downarrow \pi_2 \\ & & X_2 \end{array}$$

e.g.  $\mathcal{C} = \text{Sets}$

Q: what is  $\varprojlim (X_1, X_2)$  ?

In general we denote

$$\varprojlim (X_1, X_2) := X_1 \times X_2$$

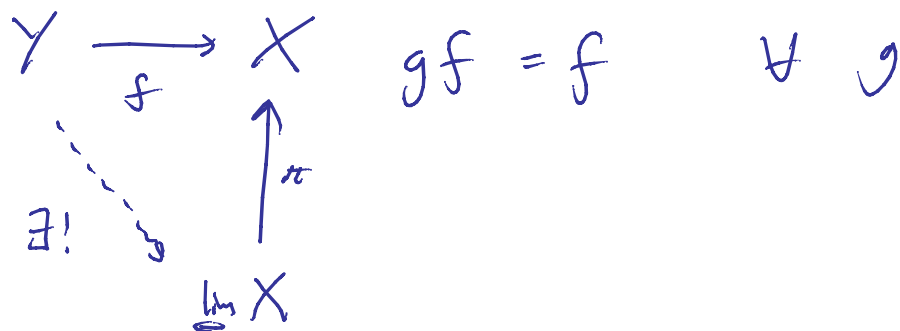
called the <sup>(categorical)</sup> product in  $\mathcal{C}$

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e.g.  $\mathcal{I} = \underline{G}$ ,  $\mathcal{C} = \text{Sets}$

$$X \in \text{Sets}^{\underline{G}}$$

$$X \supset G$$



What is  $\varprojlim X$  ?

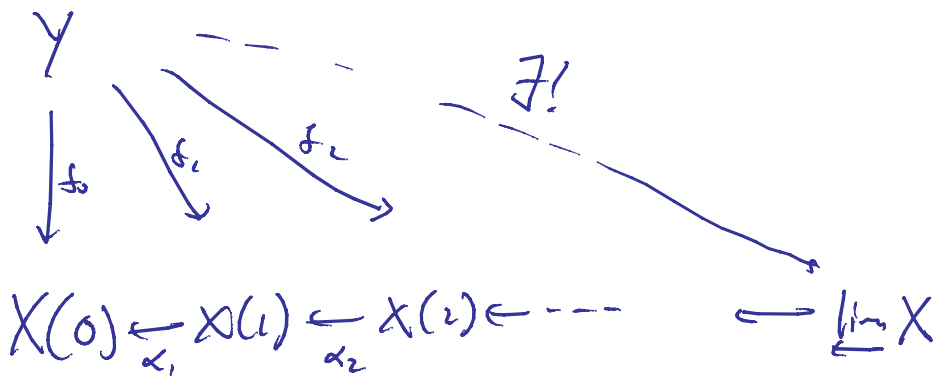
## Another important example

$$I = (0 \leftarrow 1 \leftarrow 2 \leftarrow \dots)$$

$$X \in \mathcal{C}^I$$

$$\varprojlim X$$

is called  
the projective  
limit.



e.g.  $\mathcal{C} = \text{sets}$

$$\varprojlim X = \left\{ (x_i)_{i=0}^{\infty} \mid \alpha_i(x_i) = x_{i-1} \right\}$$

e.g.

$$\varprojlim \left( \mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow \dots \right)$$

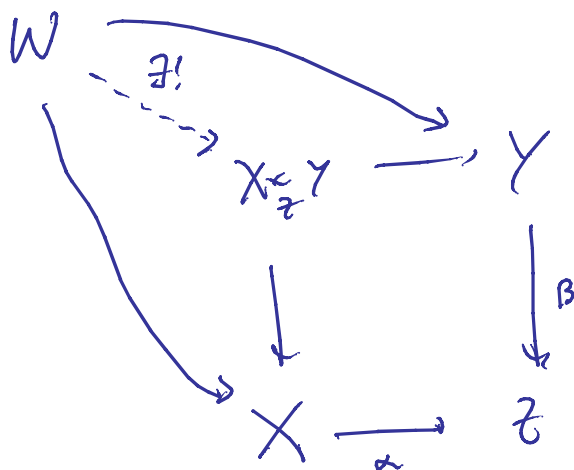
$\cong \mathbb{Z}_p^{\wedge}$

Yet another common example  
pull back

$$I = \left\{ \begin{array}{c} z \\ \downarrow \\ 1 \rightarrow 3 \end{array} \right\}$$

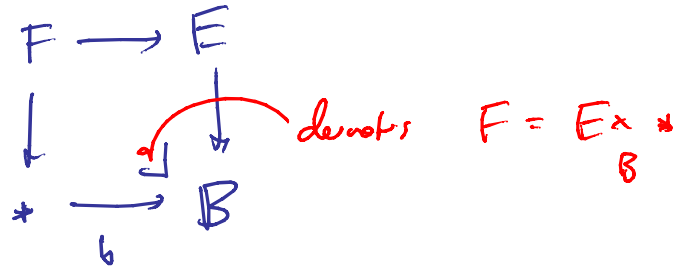
$$e^I = \left\{ \begin{array}{c} Y \\ \downarrow \\ X \rightarrow Z \end{array} \right\}$$

$\lim_{\leftarrow} \left( \begin{array}{c} Y \\ \downarrow \\ X \rightarrow Z \end{array} \right)$  is denoted  $X \times_Z Y$



e.g. in Top

$$X \times_Z Y = \left\{ (x, y) \in X \times Y \mid \alpha(x) = \beta(y) \right\} \subset_{\text{subspace}} X \times Y$$



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Comits

Same but dual