Thm: (Brown Representability)

Suppose F is an excision htpy functor.

Then $\exists$ pointed, connected CW cx $K = K_F$, unique up to htpy, and $u \in F(K)$

s.t.

$$\begin{aligned} [X, K]_* &\longrightarrow F(X) \\ f &\longmapsto f^* u \end{aligned}$$

is an isomorphism.

Rmk uniqueness of K up to htpy

(i) follows from Yoneda:

$$[-, K]_* \underset{\text{naturally iso}}{\cong} [-, K']_*$$

$\Leftrightarrow$ K isomorphic to K' in $Ho(Top_*^{CW})$

$\Leftrightarrow$ $K \simeq K'$

(ii) applying to $\tilde{H}^n(-;\pi)$

get $K = K(\pi, n)$

Rmk: This is <u>overkill</u>: Can construct $K(\pi, n)$ manually — assume π is abelian, $n \geq 2$

<u>Moore Spaces</u>:

$$\left[\bigvee_J S^n, \bigvee_I S^n\right] = \prod_J \overbrace{\left[S^n, \bigvee_I S^n\right]}^{\pi_n(\bigvee_I S^n)}$$

$$= \prod_J \bigoplus_I \underbrace{\pi_n(S^n)}_{\mathbb{Z}}$$

$$= \prod_J \mathrm{Hom}\left(\mathbb{Z}, \bigoplus_I \mathbb{Z}\right)$$

$$= \mathrm{Hom}\left(\bigoplus_J \mathbb{Z}, \bigoplus_I \mathbb{Z}\right)$$

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{\alpha} \bigoplus_I \mathbb{Z} \longrightarrow \pi \to 0$$

$$\bigvee_J S^n \xrightarrow{\tilde{\alpha}} \bigvee_I S^n \longrightarrow C(\tilde{\alpha}) =: M(\pi, n)$$

$$\tilde{H}_k(M(\pi,n)) = \begin{cases} \pi, & n=m \\ 0, & \text{o/w} \end{cases}$$

$\underline{K(\pi,n)}$ $\qquad \pi_n\, M(\pi,n) = \pi$

inductively kill higher htpy grps

$$\begin{tikzcd} \bigvee_{\pi_{n+1}(K^0)} S^{n+1} \arrow[r] & K^0 = M(\pi,n) \arrow[d] \\ \bigvee_{\pi_{n+2}(K^1)} S^{n+1} \arrow[r] & K^1 \arrow[d] \\ & K^2 \arrow[d, dotted] \\ & \vdots \end{tikzcd}$$

$$\varinjlim K^i = K(\pi,n)$$

Suppose X is a $K(\pi,n)$

$$\begin{tikzcd} \bigvee_J S^n \arrow[r, shift left] \arrow[r, shift right] & \bigvee_I S^n \arrow[r] \arrow[d] & M(\pi,n) \arrow[dl, dashed] \\ & X & \end{tikzcd}$$

$\nwarrow$ iso on π_n

Inductively

$$\begin{tikzcd} \bigvee_{\pi_{n+i}(K^{i-1})} S^{n+i-1} \arrow[r] & K^{i-1} \arrow[r] \arrow[d] & K^i \arrow[dl, dashed] \\ & X & \end{tikzcd}$$

get $K(\pi,n) \longrightarrow X$ w.e. (h.e. if X cw)

Prop: $\Omega K(\pi, n+1) \simeq K(\pi, n)$

(iii) Yoneda lemma: F, G htpy excision functors

$$\Rightarrow \quad \mathrm{Nat}(F, G) \cong [K_F, K_G]_*$$

(iv) In fact:

F, G excision htpy functors

$\eta : F \longrightarrow G$ natural transformation

and

$$\begin{array}{ccc} \eta : F(S^n) & \xrightarrow[\cong]{} & G(S^n) \\ \wr\wr & & \wr\wr \\ \pi_n(K_F) & & \pi_n(K_G) \end{array}$$

$$\Rightarrow \quad K_F \longrightarrow K_G \quad \text{w.e.}$$

$$\Rightarrow \quad \text{h.e.}$$

ie any $K(\pi, n)$ represents $\tilde{H}^n(-;\pi)$

Given (K, u) get nat transf

$$u \in F(K) \quad [-, K]_* \longrightarrow F(-)$$
$$f \longmapsto f^* u$$

(K, u) is universal if this map is an equivalence.

(pf of Brown Representability)

Def: (K,u) n-universal if

$$\pi_i(K) \longrightarrow F(S^i)$$
$$f \longmapsto f^* u$$

epi for $i \leq n$

trivial kernel for $i < n$

∞-universal

Rmk: $i \geq 1 \Rightarrow F(S^i)$ is a gp

$i > 1 \Rightarrow F(S^i)$ is an ab gp

Since $F(S^i) \times F(S^i) \cong F(S^i \vee S^i) \xrightarrow[\text{pinch}^*]{} F(S^i)$

above is a gp law.

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pointed connected CW cx

Lemma: Given (Z, z) $z \in F(Z)$

$\exists$ ∞-universal pair (K,u)

$$(Z, z) \underset{\text{subcomplex}}{\hookrightarrow} (K,u)$$

(pf) $K_1 = Z \vee \bigvee_{x \in F(S^1)} S^1$

$$F(K_1) = F(Z) \times \prod_{F(S^1)} F(S^1)$$

choose $u_1 = (z, (x)_{x \in F(S^1)})$

$$\pi_1 K_1 \twoheadrightarrow F(S^1) \qquad 1\text{-universal}$$

Inductively assume (K_n, u_n) is n-univ.

$$\ker_n \longrightarrow \pi_n(K_n) \longrightarrow F(S^n)$$

$$\bigvee_{x\in \ker_n} S^n \xrightarrow[\vee x]{} K_n \longrightarrow K_n'$$

<u>Note:</u>

$$\begin{array}{ccccc} \prod_{\ker_n} F(S^n) & \longleftarrow & F(K_n) & \longleftarrow & F(K_n') \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & & u_n & \longmapsfrom & u_n' \end{array}$$

$$K_{n+1} = K_n' \vee \bigvee_{y\in F(S^{n+1})} S^{n+1}$$

$$F(K_{n+1}) = F(K_n') \times \prod_{y\in F(S^{n+1})} F(S^{n+1})$$

$$\rotatebox{90}{$\in$}$$

$$u_{n+1} := \left(u_n', (y)_{y\in F(S^{n+1})}\right)$$

K_{n+1} obtained from K_n by attaching $(n+1)$-cells

$$\begin{tikzcd}
\alpha' \in \pi_i(K_n) \arrow[rr] \arrow[dr] & & \pi_i(K_{n+1}) \ni \alpha \arrow[dl] \\
& F(S^{n+1}) &
\end{tikzcd}$$

iso $i \leq n-1$ epi $i = n$

epi $i = n$

iso

$i < n$

epi $i = n$

iso for $i < n$

epi $i = n+1$

Suppose $\alpha \in \pi_n(K_{n+1})$

$$\alpha^* u_{n+1} = 0$$

$$\Rightarrow (\alpha')^* u_n = 0$$

$$\Rightarrow \alpha' \mapsto 0 \quad \text{in } \pi_n(K_{n+1})$$

$$\Rightarrow \alpha = 0$$

$$F(K_\infty) \longrightarrow \varprojlim F(K_n)$$

$$u \longmapsto (u_n)$$

$\square$

lemma Suppose (K, u) ∞-universal

(X, A) CW pair

given f, i, there exists $\tilde{f}$:

$$\begin{tikzcd} (A, a) \arrow[r, "f"] \arrow[d, "i"'] & (K, u) \\ (X, x) \arrow[ur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

(pf) WLOG, f is inclusion of a subcx

$Z := K \cup_A X$

MV $\Rightarrow$ get $z \in F(Z)$

embed $(Z, z) \hookrightarrow (K', u')$ $\leftarrow$ ∞-universal

$$(K, u) \longrightarrow (Z, z) \longrightarrow (K', u')$$

$$\begin{tikzcd} \pi_i(K) \arrow[rr] \arrow[dr, "\cong"'] & & \pi_i(K') \arrow[dl, "\cong"] \\ & F(S^i) & \end{tikzcd}$$

So $K \hookrightarrow K'$ is a w.e.

$\Longrightarrow$ K is a deformation retract of K'

(Compression lemma)

$$\begin{tikzcd}
(A,a) \arrow[r] \arrow[d] & (K,u) \arrow[dl] \arrow[r, equal] & (K,u) \\
(X,x) \arrow[r, hook] & (Z,z) \arrow[r, hook] & (K',u') \arrow[u]
\end{tikzcd}$$

$\square$

(pf of Brown rep)

Need to show ∞-universal $\Longrightarrow$ universal.

Suppose (K,u) ∞-universal

$$\Longrightarrow [X,K]_* \longrightarrow F(X)$$

Is this map an iso?

<u>Surjectivity</u>

$x \in F(X)$:

$$\begin{tikzcd}
(*,0) \arrow[r] \arrow[d] & (K,u) \\
(X,x) \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

✓

Injectivity

$$f, g \in [X, K]_*$$

$$f^*u = g^*u = x$$

$$\begin{tikzcd}
(X \vee X, \; (x, x)) \arrow[r, "f \vee g"] \arrow[d] & (K, u) \\
(X \wedge I_+, \; x) \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

$\square$