

11- pf of Brown Representability, $K(\mathbb{Z}, n)$ BG

Note Title

3/9/2010

Thm: (Brown Representability)

Suppose F is an excision ltpy functor.

Then \exists pointed, correct CW ∞ $K = K_F$, unique up to ltpy, and $u \in F(K)$

set,

$$[X, K]_* \longrightarrow F(X)$$

$$f \longmapsto f^* u$$

is an isomorphism.

Remark uniqueness of K up to ltpy

(i) follows from Yoneda:

$$[-, K]_* \underset{\text{is}}{\cong} [-, K']_*$$

$\Leftrightarrow K$ is isomorphic to K' in $\text{Ho}(\text{Top}^{\text{CW}})$

$$\Leftrightarrow K \simeq K'$$

(ii) apply to $\tilde{H}^n(-; \pi)$

get $K = K(\pi; n)$

Remark: This is overkill! Can construct $K(\pi, n)$ manually - assume π is abelian, $n \geq 2$

Moore Spaces:

$$\begin{aligned} [V_S^n, V_I^n] &= \prod_S [S^n, \underbrace{V_I^n}_{\pi_n(V_I^n)}] \\ &= \prod_S \bigoplus_I \underbrace{\pi_n(S^n)}_{\mathbb{Z}} \\ &= \prod_S \text{Hom}(\mathbb{Z}, \bigoplus_I \mathbb{Z}) \\ &= \text{Hom}(\bigoplus_S \mathbb{Z}, \bigoplus_I \mathbb{Z}) \end{aligned}$$

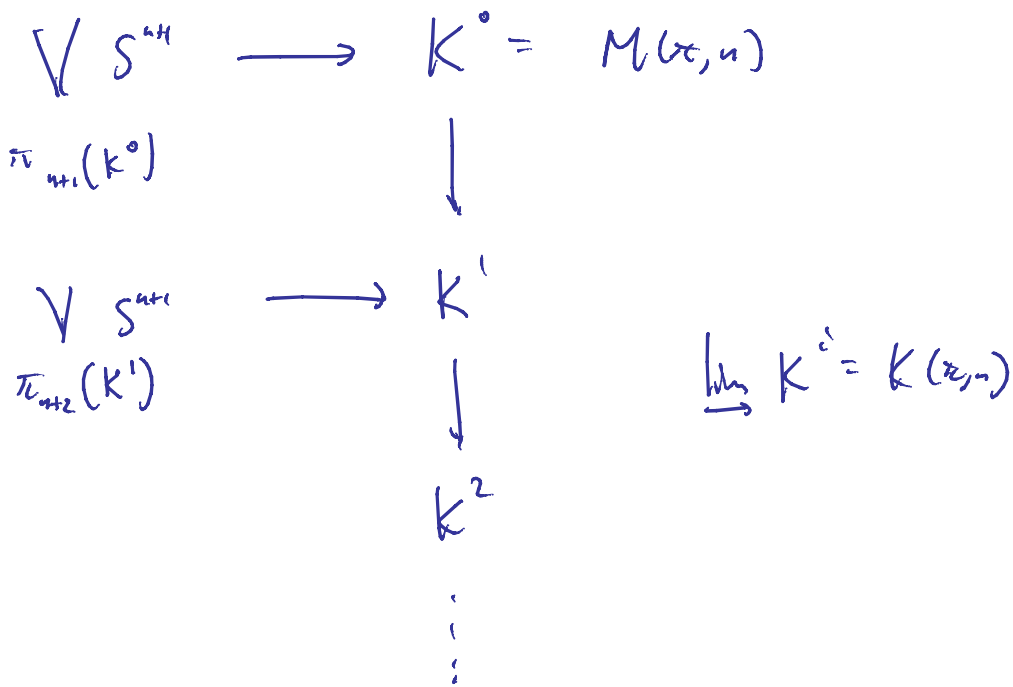
$$0 \rightarrow \bigoplus_S \mathbb{Z} \xrightarrow{\alpha} \bigoplus_I \mathbb{Z} \xrightarrow{\pi} 0$$

$$V_S^n \xrightarrow{\tilde{\alpha}} V_I^n \longrightarrow C(\alpha) =: M(\pi, n)$$

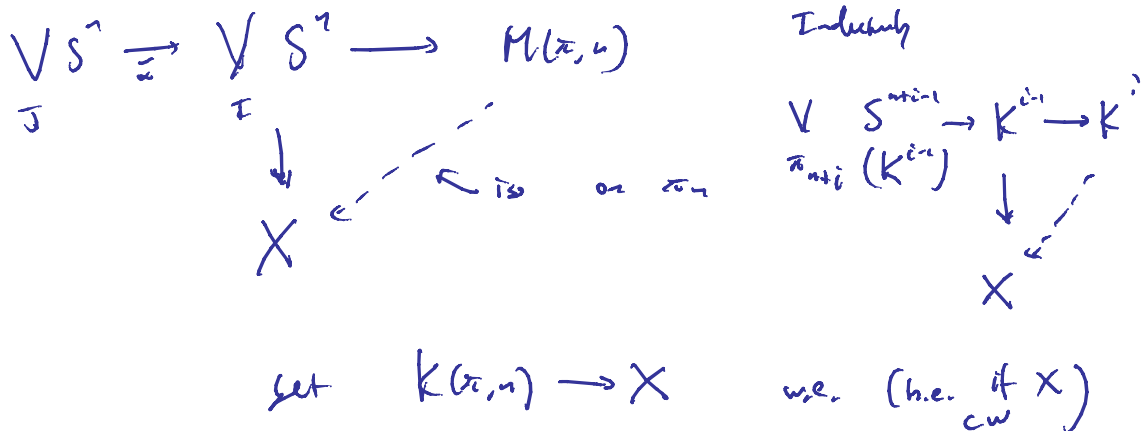
$$\tilde{H}_K(M(\pi, u)) = \begin{cases} \pi, & u = m \\ 0, & \text{o/w} \end{cases}$$

$$K(\pi, u) \quad \pi_n M(\pi, u) = \pi$$

inductively kill higher degree sgs



Suppose X is a $K(\pi, u)$



Prop' $\Omega K(\sigma, \eta) = K(\sigma, \eta)$

(iii) Yoneda - lemma: F, G are excision functors

$$\Rightarrow \text{Nat}(F, G) \cong [K_F, K_G]_*$$

(iv) In fact:

F, G excision like functors

$\eta: F \rightarrow G$ natural transformation

and

$$\begin{array}{ccc} \eta: F(S^q) & \xrightarrow{\cong} & G(S^q) \\ \parallel & & \parallel \\ \pi_1(K_F) & & \pi_1(K_G) \end{array}$$

$$\Rightarrow K_F \rightarrow K_G \quad \text{v.e.}$$

$$\Rightarrow \text{b.e.}$$

Cor any $K(\sigma, \eta)$ represents $\tilde{H}^q(-; \pi)$

Given (K, η) get out truth

$$\begin{array}{ccc} u \in F(K) & [-, z]_* \rightarrow & F(-) \\ \eta \longmapsto & & \eta^* u \end{array}$$

(K, η) is universal if this map is an equivalence.

(pf of Brown representability)

Def: (K, u) n -universal if

$$\pi_i(k) \longrightarrow F(S^i)$$

$$f \longmapsto f^* u$$

epi for $i \leq n$

trivial kernel for $i < n$

∞ -universal

Remark:

$i \geq 1 \Rightarrow F(S^i)$ is a gp

$i > 1 \Rightarrow F(S^i)$ is an ab gp

sm $F(S^i) \times F(S^i) \cong F(S^i \vee S^i) \xrightarrow{\text{pr}^*} F(S^i)$

above is a gp law.

Lemma: Given (Z, z) $z \in F(Z)$

\exists ∞ -universal pair (K, u)

$$(Z, z) \xrightarrow{\text{subcomplex}} (K, u)$$

(pf) $K_1 = Z \vee \bigvee_{x \in F(S^1)} S^1$

$$F(K_1) = F(Z) \times \prod_{F(S^1)} F(S^1)$$

choice

$$u_1 = (z, (x)_{x \in F(S^1)})$$

$$\pi_1 K_1 \longrightarrow F(S^1) \quad 1\text{-universal}$$

Inductively assume (K_n, u_n) is univ.

$$\ker_n \longrightarrow \pi_n(K_n) \longrightarrow F(S^n)$$

$$\bigvee_{x \in \ker_n} S^n \xrightarrow{\forall x} K_n \longrightarrow K_n'$$

Note:

$$\prod F(S^n) \longleftarrow F(K_n) \longleftarrow F(K_n')$$

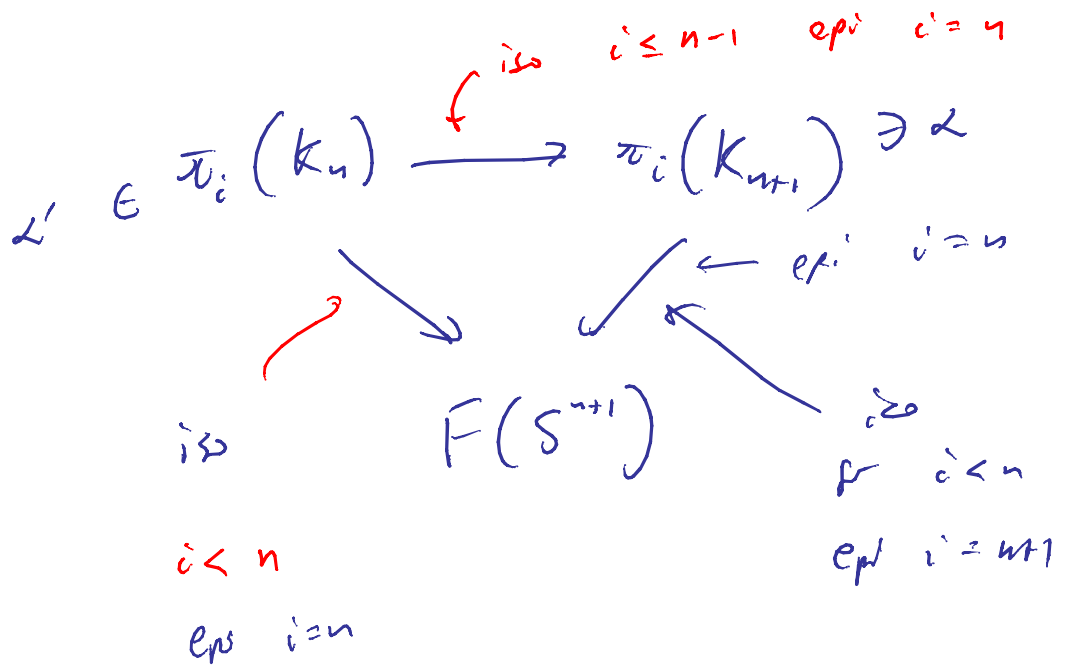
$$\ker_n \quad \cup \quad u_n \longleftarrow \cup \quad u_n'$$

$$K_{n+1} = K_n' \vee \bigvee_{\gamma \in F(S^{n+1})} S^{n+1}$$

$$F(K_{n+1}) = F(K_n') \times \prod_{\gamma \in F(S^{n+1})} F(S^{n+1})$$

$$u_{n+1} := (u_n', (\gamma)_{\gamma \in F(S^{n+1})})$$

K_{n+1} obtained from K_n by attaching $(n+1)$ -cells



Suppose $\alpha \in \pi_n(K_{n+1})$

$$\alpha \neq u_{n+1} = 0$$

$$\Rightarrow (\mathcal{L}')^\dagger u_n = 0$$

$$\Rightarrow \alpha' \mapsto 0 \text{ in } \pi_n(K_{n+1})$$

$$\Rightarrow \alpha = 0$$

$$F(K_0) \longrightarrow \varprojlim F(K_n)$$

$$u \longmapsto (u_n)$$

□

Lemma Suppose (K, u) ∞ -universal

(X, A) chr pair

given f, i , then exists \tilde{f} :

$$\begin{array}{ccc} (A, g) & \xrightarrow{f} & (K, u) \\ i \downarrow & \nearrow \tilde{f} & \\ (X, x) & & \end{array}$$

(pf) wlog, f is inclusion of a subex

$$Z := K \cup_A X$$

MV \Rightarrow let $z \in F(Z)$

embed $(Z, z) \hookrightarrow (K', u')$
 $\leftarrow \infty$ -universal

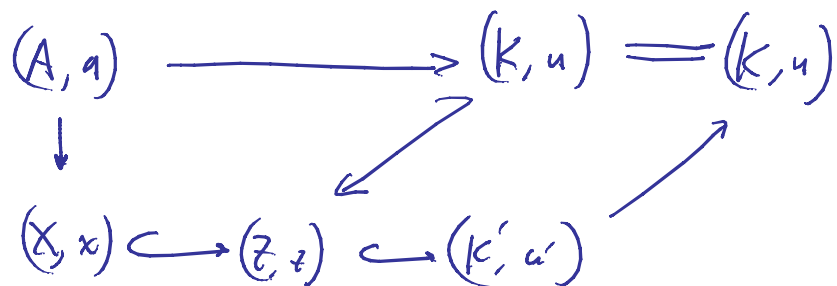
$$(K, u) \longrightarrow (Z, z) \longrightarrow (K', u')$$

$$\pi_i(k) \longrightarrow \pi_i(k')$$

$$\cong \searrow \quad \swarrow \cong \\ F(s^i)$$

So $K \hookrightarrow K'$ is a u.e.

\Rightarrow K is a deformation retract of K'
(Compression Lemma)



□

(pf of Brown rep)

Need to show ∞ -universal \Rightarrow universal.

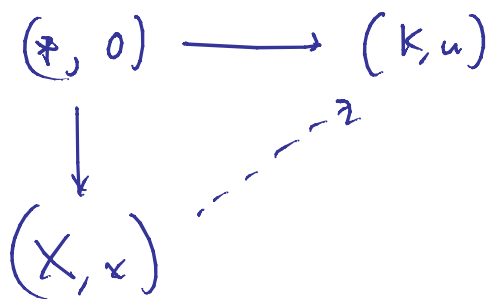
Space (K, u) ∞ -universal

$$\Rightarrow [X, K]_* \rightarrow F(X)$$

Is this map an iso?

Surjectivity

$x \in F(X) :$



✓

Injective

$$f, g \in [X, K]_*$$

$$f^*u = g^*u = x$$

$$\begin{array}{ccc} (X \vee X, (x, x)) & \xrightarrow{f \vee g} & (K, u) \\ \downarrow & \nearrow H & \\ (X \sim I_+, x) & & \end{array}$$

□
