12 - Classifying spaces

Note Title 3/9/2010

Principle G-bundles

$G =$ top'l gp

> **Rmk:** $X \to Y$ pointed w.e.
>
> $$\begin{tikzcd} X \arrow[r] \arrow[dr, hook] & Y \\ & \overline{Cyl}(f) \arrow[u] \end{tikzcd}$$
>
> def retract by compression thm
>
> pointed w.e.

A **G-space**

$$G \times X \xrightarrow[\mu]{f} X \quad \text{continuous}$$

$$\mu(1, x) = x$$

$$\mu(gh, x) = \mu(g, \mu(h, x))$$

A **principle G-bundle** $/B$

$$\begin{array}{l} E \text{ w/ } G\text{-space} \\ \downarrow p \\ B = E/G \end{array}$$

s.t. $\forall\ x \in B$ $\exists$ nbhd U of x

equivalent homeo

$$\begin{tikzcd} G \times U \arrow[r, "\approx"] \arrow[dr, "\pi_2"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

iso:

$$\begin{tikzcd} E \arrow[rr, "f\ \text{equivariant}", "\cong"'] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Rmk: E, E' prin. G-bdls $/B$

any equivariant map $E \to E'$ is an iso.

G-bundles & free G-CW complexes:

lem $G\times - : \mathrm{Top} \rightleftarrows \mathrm{Top}_G$: forget

is an adjoint pair of functors

i.e.
$$\begin{array}{ccc} G\times X \longrightarrow Z & & G\text{-map} \\ \updownarrow & & \\ X \longrightarrow Z & & \text{map} \end{array}$$

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lem:
$$\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$$
princ G-bundle must be trivial

(pf) trivialization $\longleftrightarrow$ section

$$\begin{tikzcd} & E \arrow[d] \\ G\times D^n \arrow[ur] \arrow[r] & D^n \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} & E \arrow[d] \\ D^n \arrow[ur] \arrow[r, equal] & D^n \end{tikzcd}$$

exists since

$$\begin{tikzcd} * \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \arrow[r, equal] \arrow[ur, dashed] & D^n \end{tikzcd}$$

$\begin{array}{c} E \\ \downarrow \\ D^n \end{array}$ is a serre fibration.

$\square$

Def: A free G-CW cx. is a G-space

$$Z = \varinjlim Z^{[n]}$$

$$\begin{tikzcd}
\coprod G \times S^{n-1} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & Z^{[n-1]} \arrow[d] \\
\coprod G \times D^n \arrow[r] & Z^{[n]}
\end{tikzcd}$$

Prop: $\begin{matrix} E \\ \downarrow p \\ B \end{matrix}$ prin. G-bundle, B = CW cx

$\Rightarrow$ E is a free G-CW cx

Pf:

$$\begin{tikzcd}
\coprod S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B^{[n-1]} \arrow[d] \\
\coprod D^n \arrow[r, "\sqcup \alpha_i"'] & B^{[n]}
\end{tikzcd}$$

$\alpha_i^* E|_{B^{[n]}}$ trivial $\Longrightarrow$

$$\begin{tikzcd}
\coprod G \times S^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E|_{B^{[n-1]}} \arrow[d] \\
\coprod G \times D^n \arrow[r] & E|_{B^{[n]}}
\end{tikzcd}$$

$\square$

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array} \quad \text{is pointed if} \quad * \in B, \; e \in p^{-1}(*)$$

$$\begin{array}{c} e \longrightarrow E \\ \updownarrow \\ G \underset{G\text{-map}}{\longrightarrow} E \end{array}$$

Consider: $F : (\mathrm{Top}_*^{cw})^{op} \longrightarrow \mathrm{Set}_*$

$$F(X) = \left\{ \text{iso classes of pointed principal } G\text{-bundles} / X \right\}$$

pointed w/ trivial G-bundle

Functoriality

$$\begin{tikzcd} f^*E = E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ f : X \arrow[r] & Y \end{tikzcd}$$

Claim: F is a htpy functor

$$K_F =: BG$$

(i) **Homotopy** $H : f \simeq f'$

$$\begin{tikzcd} H^*E \arrow[r] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r, "\simeq"', "\text{pointed htpy}"'] & Y \end{tikzcd}$$

Claim:

$$H^*E|_{X \times \{0\}} \cong H^*E|_{X \times \{1\}}$$

$$\begin{array}{ccc} \shortparallel & & \shortparallel \\ E_0 & & E_1 \end{array}$$

Claim:

$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d, hook] & E_0 \arrow[d] \\
H^*E_0 \arrow[r] \arrow[ur, dashed] & X \times I \arrow[r] & X
\end{tikzcd}$$

in cat Top_q

(Then $E_1 \to H^*E_0 \to E_0$ is the desired iso)

Induction

$$\begin{tikzcd}
\coprod D^n \times 0 \cup S^{n-1} \times I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \times \{0\} \cup X^{[n-1]} \times I \arrow[d] \\
\coprod D^n \times I \arrow[r] & X \times \{0\} \cup X^{[n]} \times I
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod G \times (D^n \times 0 \cup S^{n-1} \times I) \arrow[r] \arrow[d] & E_0 \cup H^*E|_{X^{[n-1]} \times I} \arrow[d, hook] \arrow[ddr, bend left, "f_{n-1}"] \\
\coprod G \times D^n \times I \arrow[r] \arrow[d] \arrow[drr, dashed] & E_0 \cup H^*E|_{X^{[n]} \times I} \arrow[dr, dashed] \\
\coprod D^n \times I \arrow[rr, bend right] & & E_0 \arrow[d] \\
& & X
\end{tikzcd}$$

f_{n-1} ← inductively

$\updownarrow$

$$\begin{tikzcd}
\coprod D^n \times 0 \cup S^{n-1} \times I \arrow[r] \arrow[d, "\leftarrow \text{deformation retract}"] & E_0 \cup H^*E|_{X^{[n-1]} \times I} \arrow[r] & E_0 \arrow[d] \\
\coprod D^n \times I \arrow[urr, dashed] \arrow[rr, bend right] & & X
\end{tikzcd}$$

this exists since $\begin{matrix} E_0 \\ \downarrow \\ X \end{matrix}$ is a Serre fib D

(ii) M-V axiom $\quad Z = X \cup_A Y$

$$E/_X, \quad E'/_Y \qquad E|_A \simeq E'|_A$$

$$\begin{tikzcd} E|_A \simeq E'|_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & E' \arrow[d] \\ E \arrow[r] & E'' \end{tikzcd} \qquad E''/_G \simeq Z$$

(iii) wedge

$$\begin{tikzcd} F_\alpha \arrow[d] \\ B_\alpha \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \bigcup E_\alpha \arrow[d] \\ \bigvee B_\alpha \end{tikzcd}$$

identify basepointed fibers

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pointed conn cx $\quad BG$

Brown Rep. $\Longrightarrow \exists$

s.t. $\quad [X, BG]_* \simeq \{\text{pointed } G\text{-bdles}/X\}$

$$BG \xrightarrow[\mathrm{id}]{} BG \quad \rightsquigarrow \quad \begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$$

"universal G-bundle"
