

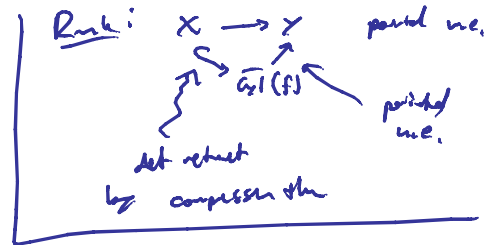
12 - Classifying spaces

Note Title

3/9/2010

Principle G -bundles

$$G = \text{top'l } \text{gp}$$



A G -space

$$G = X \xrightarrow[\mu]{f \text{ constant}} X$$

$$\mu(L, x) = x$$

$$\mu(gx, x) = \mu(g, \mu(L, x))$$

A principle G -bundle / B

E \leftarrow G -space

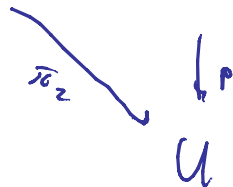
$\downarrow p$

$$B = E/G$$

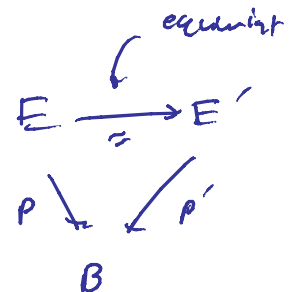
s.t. $\forall x \in B$

\exists U s.t. U of x

$G \times U \xrightarrow{\delta \text{ equivariant homeo}} p^{-1}(U)$



iso



E, E' princ. G -bundles / B

Rank:

any equivariant map $E \rightarrow E'$
is an iso.

G-bundles + free G-CW complexes:

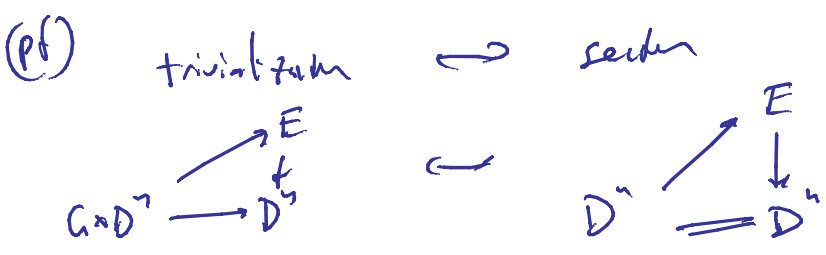
lem $G \times - : \text{Top} \rightleftarrows \text{Top}_G : \text{forget}$

is an adjoint pair of functors

i.e.

$$\begin{array}{ccc}
 G \times X & \longrightarrow & Z & G\text{-map} \\
 & & \updownarrow & \\
 X & \longrightarrow & Z & \text{map}
 \end{array}$$

lem: $E \downarrow D^n$ prime G-bundle must be trivial



Def: A free G -CW ex.
 is a G -space
 $Z = \varinjlim Z^{[n]}$

$$\begin{array}{ccc} \downarrow G \times S^{n-1} & \longrightarrow & Z^{[n-1]} \\ \downarrow \Gamma & & \downarrow \\ \downarrow G \times D^n & \longrightarrow & Z^{[n]} \end{array}$$

Prop: E free G -module, $B = \text{CW ex.}$
 $\downarrow P$
 B

$\Rightarrow E$ is a free G -CW ex.

(P1)

$$\begin{array}{ccc} \downarrow S^{n-1} & \longrightarrow & B^{[n-1]} \\ \downarrow \Gamma & & \downarrow \\ \downarrow D^n & \xrightarrow{\downarrow \alpha_i} & B^{[n]} \end{array}$$

$\alpha_i^* E|_{B^{[n]}}$ trivial \Rightarrow

$$\begin{array}{ccc} \downarrow G \times S^{n-1} & \longrightarrow & E|_{B^{[n-1]}} \\ \downarrow \Gamma & & \downarrow \\ \downarrow G \times D^n & \longrightarrow & E|_{B^{[n]}} \end{array}$$

□

E
 $\downarrow p$
 B is pointed if $* \in B, e \in p^{-1}(*)$

$$\begin{array}{ccc} e & \longrightarrow & E \\ & \downarrow & \\ G & \longrightarrow & E \\ & \text{Group} & \end{array}$$

Consider: $F : (\text{Top}_*^{CW})^{op} \rightarrow \text{Set}_*$

$$F(X) = \left\{ \begin{array}{l} \text{is} \\ \text{of} \\ \text{pointed} \end{array} \begin{array}{l} \text{classes} \\ \text{of} \\ \text{principal} \\ \text{bundles} \end{array} (X) \right\}$$

Functoriality

$$\begin{array}{ccc} \mathcal{F}^* E & & \\ \text{"} & & \\ E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathcal{F} X & \longrightarrow & Y \end{array}$$

pointed w/ trivial G-bundle

Claim: F is a topological functor
 $K_F =: BG$

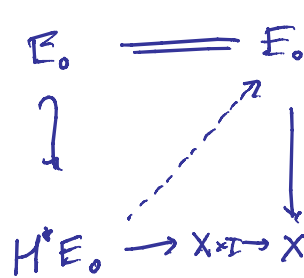
(i) Monotopy $H: \mathcal{F} \simeq \mathcal{F}'$

$$\begin{array}{ccc} H^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & Y \\ & \text{2 pointed} & \\ & \text{topology} & \end{array}$$

Claim:

$$\begin{array}{ccc} H^* E /_{X \times \{0\}} & \simeq & H^* E /_{X \times \{1\}} \\ \text{"} & & \text{"} \\ E_0 & & E_1 \end{array}$$

Claim:



(Then $E_1 \rightarrow H^*E_0 \rightarrow E_0$ is the desired iso)

Induction

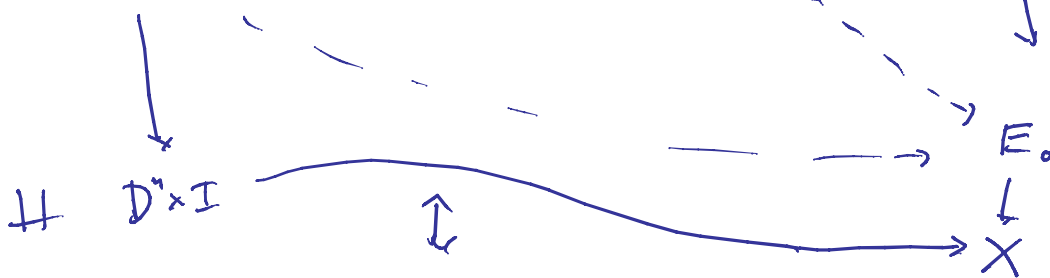
in cat Top_G

$$\coprod D^2 \times 0 \cup S^{n-1} \times I \longrightarrow X \times \{0\} \cup X^{(n-1)} \times I$$

$$\downarrow \quad \quad \quad \downarrow \\
 \coprod D^2 \times I \longrightarrow X \times \{0\} \cup X^{(n)} \times I$$

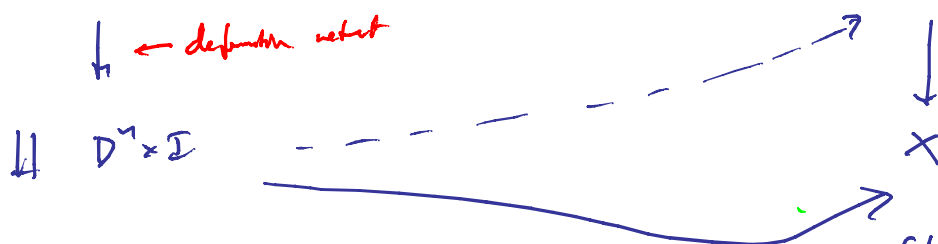
$$\coprod G_n(D^2 \times 0 \cup S^{n-1} \times I) \longrightarrow E_0 \cup H^*E|_{X^{(n-1)} \times I}$$

$$\downarrow \quad \quad \quad \downarrow \\
 \coprod G_n(D^2 \times I) \longrightarrow E_0 \cup H^*E|_{X^{(n)} \times I}$$



f_{n-1} *inductively*

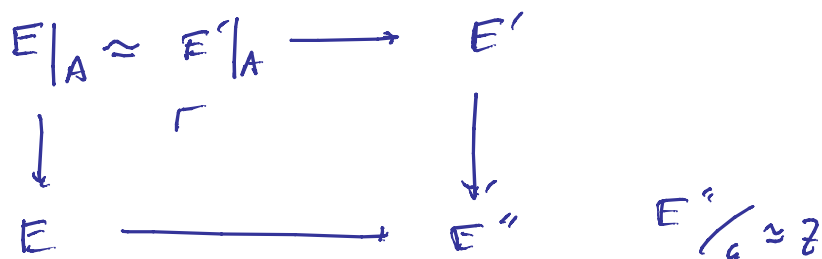
$$\coprod D^2 \times 0 \cup S^{n-1} \times I \longrightarrow E_0 \cup H^*E|_{X^{(n-1)} \times I} \rightarrow E_0$$



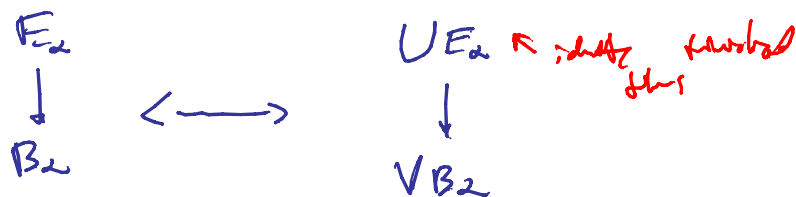
this exists since $\frac{E_0}{X}$ is a Serre fib D

(ii) M-V axiom $Z = X \cup_A Y$

$E/x, E/y \quad E|_A \approx E'|_A$



(iii) Weak



Brown Rep. $\implies \int$ point for cos B_G

Set. $[x, B_G]_* \cong \{ \text{pointed } G\text{-objs } / x \}$

