

14 - Spectral sequences

Note Title

3/30/2010

Problem: $F \rightarrow E \rightarrow B$ fiber sequence

(i.e. $E \rightarrow B$ some fibration, fiber F)

What is relationship between

$$\begin{array}{l} H_*(F) \\ H_*(E) \quad ? \\ H_*(B) \end{array}$$

A: Serre Spectral Sequence

$$E_2^{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E)$$

(we will explain)

Idea! $B = \text{CW } \infty$

$$E = \varinjlim E^{[s]}$$

$\downarrow p$

$$B = \varinjlim B^{[s]}$$

$$E^{[s]} = p^{-1}(B^{[s]})$$

Get a filtration: $C_+^{\text{sing}}(E) = \varinjlim C_+^{\text{sing}}(E^{[s]})$

Problem: $C_* = \lim_{\leftarrow} (F_0 C_* \hookrightarrow F_1 C_* \hookrightarrow F_2 C_* \hookrightarrow \dots)$
 filtered chain complex

What is relationship between

$$\bigoplus_s H_* \left(\frac{F_s C_*}{F_{s+1} C_*} \right) \quad \text{and} \quad H_* C_* \quad ?$$

A: Spectral sequence of a filtered complex

$$E'_{s,t} = H_{s+t} \left(\frac{F_s C_*}{F_{s+1} C_*} \right) \Rightarrow H_{s+t} C$$

Def: A spectral sequence (of homological type)

$$\{E_{s,t}^r\}$$

is a sequence of bigraded abelian groups

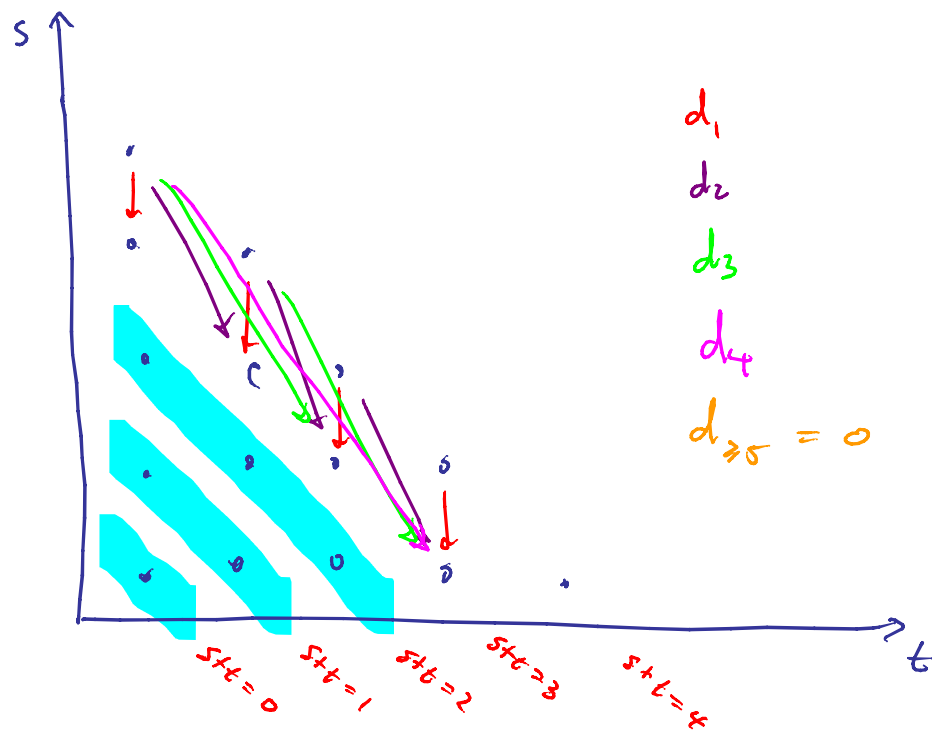
$$\left(\text{typically } r \geq 1, \underbrace{s, t \geq 0}_{\text{first quadrant}} \right)$$

equipped w/ maps
 "first quadrant" (we will always assume)

$$d_r : E_{s,t}^r \rightarrow E_{s-r, t+r-1}^r$$

s.t. $d_r^2 = 0$

$$E_{s,t}^{r+1} = H_{s,t}(E_{s,t}^r, d_r)$$



- $d_r : (s+t) \longrightarrow (s+t) - 1$

- In a particular diagonal, only finitely many d_r 's are non-zero

For fixed (s,t)

- $E_{s,t}^n = E_{s,t}^{n+1} \quad n \gg 0$

we call this $E_{s,t}^\infty$

Notation! let A_n be a graded ab gp

we say: $E_{s,t}^n \implies A_{s,t}$

If there exists a filtration

$$F_0 A_n \subseteq F_1 A_n \subseteq F_2 A_n \subseteq \dots \subseteq A_n$$

St • $\lim_s F_s A_n = A_n$

• $E_{s,t}^\infty \cong \frac{F_s A_{st}}{F_{s-1} A_{st}} \} Gr_s A_{st}$

" A_n built out of $Gr_s A_n$ "

In our case: $Gr_s C_* := \frac{F_s C_*}{F_{s-1} C_*}$

$E'_{s,t} = H_{st}(Gr_s C_*) \Rightarrow H_{st} C_*$

$E_{s,t}^\infty = Gr_s H_{st} C_*$

Slogan! "Spectral sequence commutes Gr_* past H_* "

Construct: "assemble" LES's

$\dots \rightarrow H_n(F_{s-1}C) \rightarrow H_n(F_s C) \rightarrow H_n\left(\frac{F_s C}{F_{s-1} C}\right) \xrightarrow{d} H_{n-1}(F_{s-1}C) \rightarrow \dots$

$$E_{s,t}^r = \frac{\widehat{Z}_{s,t}^r}{B_{s,t}^r} \quad \text{Inductively:}$$

$$\widehat{Z}_{s,t}^r = \partial^{-1} \left(\text{Im} \left(H_{s+t-1}(F_{s-r}) \rightarrow H_{s+t-1}(F_{s-t}) \right) \right)$$

$$B_{s,t}^r = \text{Im} \left(H_{s+t}(F_s) \rightarrow H_{s+t}(F_{s+r-1}) \right)$$

Inductively check d_r exists and is well defined.

Note:

$$d_i: H_{s+t}(F_s/F_{s-1}) \rightarrow H_{s+t-1}(F_{s-1}) \rightarrow H_{s+t-1}(F_{s-1}/F_{s-2})$$

We think of $E_{s,t}^s$ as a subquotient
of $E_{s,t}^1$

$$x \in E_{s,t}^1$$

$$x \in \widehat{Z}_{s,t}^r \iff d_i(x) = 0 \quad i < r$$

$$x \in B_{s,t}^r \iff x \in \text{im } d_i \quad \text{some } i < r$$

$$x \in \widehat{Z}_{s,t}^r \iff d_r(x) = 0 \quad \forall r$$

" x is a permanent cycle"

$$\begin{aligned}
 x \text{ P.C.} &\implies \partial(x) = 0 \\
 &\implies x = g(x')
 \end{aligned}$$

$$\begin{array}{ccc}
 H_{stt}(F_s) & \longrightarrow & H_{stt}(C) \\
 \varphi \downarrow & & \downarrow \\
 x' & \longmapsto & \tilde{x}
 \end{array}$$

Defn a filtration on $H_{stt}(C)$

$$F_s H_{stt}(C) := \text{Im} \left(H_{stt}(F_s) \rightarrow H_{stt}(C) \right)$$

$$\implies \tilde{x} \in F_s H_{stt}(C)$$

$$x = g(x')$$

$$\implies x' - x'' \in \ker(g)$$

$$\tilde{x} - \tilde{x}' \in F_{s-1} H_{stt}(C)$$

So get well defined map:

$$\begin{array}{ccc}
 Z_{s,t}^{\infty} & \longrightarrow & \frac{F_s H_{stt}(C)}{F_{s-1} H_{stt}(C)} \\
 \varphi \downarrow & & \downarrow \\
 x & \longmapsto & \tilde{x}
 \end{array}
 \implies E_{s,t}^{\infty} = \frac{F_s H_{stt}(C)}{F_{s-1} H_{stt}(C)}$$

kernel = B_{st}^{∞}