15 Serre Spectral Sequence

Note Title 4/1/2010

$$F \longrightarrow E \longrightarrow B$$

Serre fiber sequence
B connected

Serre spectral sequence:

$$E^2_{s,t} = \underbrace{H_s(B; H_t(F))}_{\text{local coef!}} \Rightarrow H_{s+t}(E)$$

[Do you know what this means]

$\pi_1 B = 0 \Rightarrow$ no issues

Construction

$$E = \varinjlim E^{[s]} \qquad \underbrace{C_*^{sing}(E) = \varinjlim C_*^{sing}(E^{[s]})}_{\text{filtered cx:}}$$

$$\downarrow$$

$$B = \varinjlim B^{[s]}$$

SS filtered cx

$$\Rightarrow \quad E^1_{s,t} = \underbrace{H_{s+t}\left(C_\bullet^{sing}(E^{[s]}) / C_\bullet^{sing}(E^{[s-1]})\right)}_{H_{s+t}(E^{[s]}, E^{[s-1]})} \Rightarrow \underbrace{H_{s+t}(C_\bullet^{sing} E)}_{H_{s+t} E}$$

All that's left is to identify E^2_{st} ...

Aside on twisted cohomology

X, connected CW cx.

Coefficient system $=$ functor $\pi_{oid} X \longrightarrow Ab$

$$x \longmapsto M_x$$

$$x \xrightarrow{\gamma} y \qquad \gamma_* : M_x \longrightarrow M_y$$

M is determined by $M_x \circlearrowleft \pi_1(X,x)$

Pick $\forall\, y$, $x \xrightarrow[\gamma_y]{} y$

these give isos $M_x \xrightarrow[\cong]{} M_y$

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma_y"'] & y \end{tikzcd} \qquad \begin{tikzcd} M_x \arrow[r, bend left, "\gamma"] \arrow[dr, dashed, "(\gamma_y^{-1}\gamma)_*"'] & M_y \arrow[d, phantom, "\wr ?"] \\ & M_x \end{tikzcd}$$

in $\pi_1(X,x)$

<u>Note</u>: $f : X \to Y \implies$ $M =$ coef sys on Y
$f^*M =$ coef sys on X

$$f^*M : \pi_{oid} X \xrightarrow[f_*]{} \pi_{oid}\, Y \xrightarrow[M]{} Ab$$

Two different perspectives:

$\forall$ s-cells e_i^s of X, pick $x_i \in$ interior e_i^s

$e_i^s : D^s \longrightarrow X$

$(e_i^s)^* M$ = coef sys on D^s $\longleftrightarrow$ ab gp $M_{e_i^s} = M_{x_i}$

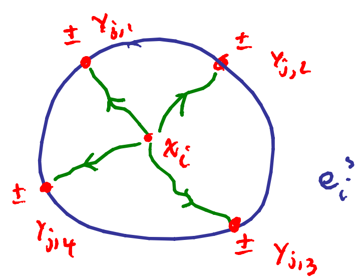
$$\begin{tikzcd} \partial D_i^s \arrow[r, "\partial e_i^s"] \arrow[rrr, bend right, "\alpha_{ij}"'] & X^{[s-1]} \arrow[r] & \bigvee S_j^{s-1} \arrow[r] & S_j^{s-1} \end{tikzcd}$$

$\alpha_{ij} \simeq \alpha_{ij}'$ map w/ regular value x_j'

$M_{x_j'} \cong M_{e_j^{s-1}}$

$(\alpha_{ij}')^{-1}(x_j') = \{y_{j,k}\} \subset \partial D_i^s$

pick γ_{ik}



Form: "twisted cellular chain cx"

$$\cdots \to \bigoplus_{i \in s\text{-cells}} M_{x_i} \xrightarrow{\partial} \bigoplus_{j\text{-}(s-1)\text{-cells}} M_{x_j} \to \cdots$$

$m \in M_{x_i}$

$$\partial(m) = \sum_j \sum_k (\gamma_{i,j})_*(m)$$

Alternatively:

$$\begin{array}{c}\tilde{X} \\ \downarrow \\ X\end{array} \leftarrow \text{universal cover} \qquad \text{Regard } M \text{ as } \pi_1\text{-module}$$

$$s\text{-cells}(\tilde{X}) \circlearrowleft \pi_1(X) \qquad \text{free action}$$

$$C_*^{cell}(\tilde{X}) = \text{free } \mathbb{Z}[\pi_1(X)]\text{-module}$$

trivial π_1-module

$$C_*^{cell}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z} \cong C_*^{cell}(X).$$

$$H_*\left(C_*^{cell}(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} M\right) = H_*^{twisted}(X;M)$$

Note: $M = \mathbb{Z} \Rightarrow$ regular H_*

Trivializations and π_1-action:

lemma: $E \to B$ Serre fib, X CW cx

$$\Rightarrow \quad \begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & E \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

"Serre fibrations behave like fibrations for CW cx's"

(pf) Induction up $X^{[s]}$

$$\begin{tikzcd} I^s \arrow[d, hook] & \approx & D^s \times \{0\} \cup \partial D^s \times I \arrow[r] \arrow[d] & E \arrow[d] \\ I^{s+1} & \approx & D^s \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

$\square$

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lemma:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ I^s \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} F' \times I^s \arrow[rr, dashed, "w.e."] \arrow[dr, "\pi_2"'] & & E' \arrow[dl, "p'"] \\ & I^s & \end{tikzcd}$$

$F' =$ fiber over $(0, \dots, 0)$
$\in I^s$

(Pf)

LES

$$\begin{tikzcd} F' \times \{0\} \arrow[r, "\sim"] \arrow[d, "\sim"'] & E' \arrow[d] \\ F' \times I \arrow[r] \arrow[ur, dashed] & I \end{tikzcd}$$

So this is equiv.

If F' is a CW cx

If not, have to CW approximation...

"weak parallel transport" $b \xrightarrow{\gamma} b'$

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ I \arrow[r, "\gamma"'] & B \end{tikzcd}$$

get:

$$\begin{tikzcd} F_b \times \{0\} \arrow[r] \arrow[d] & E' \arrow[d] \\ F_b \times I \arrow[r] \arrow[ur, dashed, "\sim"] & I \end{tikzcd}$$

gives $F_b \underset{\sim}{\xrightarrow{\gamma_*}} F_{b'}$

Different perspective

$$\begin{tikzcd} & F_{b'} \arrow[d, "\sim"] \\ F_b \arrow[r, "\sim"'] & E' \end{tikzcd} \qquad \begin{tikzcd} & H_* F_{b'} \arrow[d, "\cong"] \\ H_* F_b \arrow[r, "\cong"'] \arrow[ur, dashed, "\gamma_*"] & H_* E' \end{tikzcd}$$

this map independent of choice of lift

Lem: $\gamma \simeq \gamma'$ rel $\partial I \implies \gamma_* = \gamma'_* : H_* F_b \longrightarrow H_* F_{b'}$

(pf)
$$\begin{tikzcd} H^* E \arrow[r] \arrow[d] & E \arrow[d] \\ I \times I \arrow[r, "h"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & \gamma^* E \arrow[d, "\simeq_H"] & \\ F_b \arrow[ur, "\simeq_H"] \arrow[dr, "\simeq_H"'] \arrow[rr, bend left=60, "\gamma_*"] \arrow[rr, bend right=60, "\gamma'_*"'] & h^* E & F_{b'} \arrow[ul, "\simeq_H"'] \arrow[dl, "\simeq_H"] \\ & (\gamma')^* E \arrow[u, "\simeq_H"'] & \end{tikzcd}$$

apply H_*

Action of π_1 $\quad \gamma \in \pi_1(B)$

$$\Longrightarrow \gamma_* : H_*(F) \longrightarrow H_*(F)$$

check: • independent of htpy class of γ

• gives action.

Identification of E_2

$\forall$ cells $D^s \xrightarrow[e_i^s]{} B$

pick $b_i \in \partial D^s$, trivialization: $D^s \times F_{b_i} \dashrightarrow E|_e$, with maps $D^s \times F_{b_i} \to D^s$ and $E|_e \to D^s$

$$E^1_{s,t} = H_{s+t}(E^s, E^{s-1})$$

$$\cong \tilde{H}_{s+t}\left(\coprod_i E|_{e_i^s}, \coprod_i E|_{\partial e_i^s}\right)$$

careful excision argument

$$\cong \bigoplus_i \tilde{H}_{s+t}\left(F_{b_i} \times D^s, F_{b_i} \times \partial D^s\right)$$

$$\cong \bigoplus_i \tilde{H}_{s+t}\left(F_{b_i +} \wedge D^s/\partial D^s\right)$$

$$\underset{\text{Kunneth}}{\cong} \bigoplus_i H_t(F_{b_i}) \otimes \tilde{H}_s(D^s/\partial D^s)$$

$$d_1 : H_{s+t}(E^s, E^{s-1}) \longrightarrow H_{s+t-1}(E^{s-1}) \longrightarrow H_{s+t-1}(E^{s-1}, E^{s-2})$$

$$\begin{array}{ccc} \cong & & \cong \\ \bigoplus_{i \in s\text{-cells}(B)} H_t(F_{b_i}) & & \bigoplus_{j \in (s-1\text{-cells})} H_t(F_{b_j}) \end{array}$$

$$\Longrightarrow \quad E^2_{s,t} = H_s(B; H_t(F))$$
