

19 - Chern classes

Note Title

4/13/2010

Recall that a characteristic class ξ of n -plane bundles (real, $\mathbb{C}\mathbb{X}$) is a natural transformation

$$\xi: \text{Vect}_n(-) \longrightarrow H^{\xi}(-; \mathbb{R})$$

(i.e.) $\begin{array}{c} V \\ \downarrow \\ X \end{array} \longrightarrow \xi(V) \in H^{\xi}(X; \mathbb{R})$

$$X \xrightarrow{f} Y \implies \xi(f^*V) = f^*(\xi(V))$$

$$\left. \begin{array}{l} \text{Vect}_n^{\mathbb{C}}(X) = [X, BU(n)] \\ H^{\xi}(-; \mathbb{R}) = [X, K(\mathbb{R}, i)] \end{array} \right\} \xrightarrow{\text{Yoneda}} \xi \in [BU(n), K(\mathbb{R}, i)] \cong H^{\xi}(BU(n); \mathbb{R})$$

Recall $\begin{array}{ccc} V & \longrightarrow & EU(n) \times_{U(n)} \mathbb{C}^n = V_{\text{univ}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BU(n) \end{array}$

$$V = f^* V_{\text{univ}}$$

Moral: {characteristic classes} \longleftrightarrow $\{H^* BU(n)\}$

e.g. $BU(1) = K(\mathbb{Z}, 2) = \mathbb{C}P^{\infty}$

$$H^*(BU(1)) = \mathbb{Z}[c_1] \quad |c_1| = 2$$

gives $\begin{array}{c} L \\ \downarrow \\ X \end{array}$ the bundle, $c_1(L) \in H^2(X; \mathbb{Z})$

$$c_1 : BU(1) \xrightarrow{\cong} K(\mathbb{Z}, 2)$$

$\Rightarrow L$ is determined by $c_1(L)$

Thm: There exist characteristic classes

$$c_i(V) \in H^{2i}(X; \mathbb{Z}) \quad \text{for all } n \times n \text{ v.b.'s } V$$

$$1 \leq i \leq n$$

Convention: $c_0(V) = 1 \in H^0(X)$
 $c_i(V) = 0, \quad i > n$

Satisfy

Cartan Formula:

$$c_i(V \oplus W) = \sum_{i_1 + i_2 = i} c_{i_1}(V) c_{i_2}(W)$$

Rank: special case of Cartan:

$$c_i(V \oplus \mathbb{R}) = c_i(V) \cdot \underbrace{c_0(\mathbb{R})}_1 + c_{i-1}(V) \cdot \underbrace{c_1(\mathbb{R})}_1$$

$$= c_i(V)$$

In particular $c_i(\mathbb{R}^n) = 0 \quad \forall i > 0$

Product the classes

Thm $H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \rightsquigarrow \begin{array}{ccc} V & \xrightarrow{\quad} & V_{\text{new}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BU(n) \end{array} \rightsquigarrow c_n(V) = f^*(c_n)$$

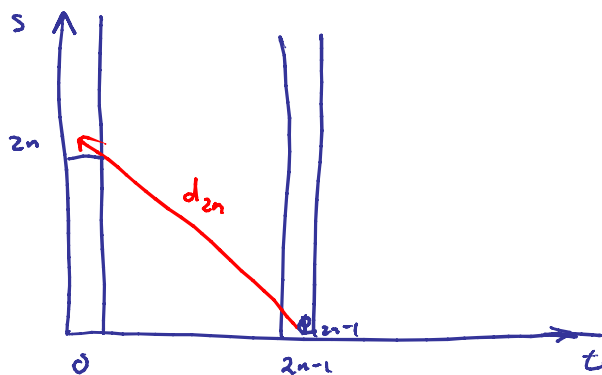
(pf of Thm)

Induction on n ($n=1$ ✓)

$$\begin{array}{c} U(n) / U(n-1) \longrightarrow BU(n-1) \longrightarrow BU(n) \\ \parallel \\ \cong^{2n-1} \end{array}$$

SSS

$$H^*(BU(n)) \otimes \Lambda[e_{2n-1}] \Rightarrow H^*BU(n-1)$$



deduce:

$$H^i(BU(n)) \xleftarrow{\cong} H^i(BU(n-1))$$

for $i < 2n-1$

Define $d_{2n}(e_{2n-1}) = c_n \longleftarrow$ (Note: since $H^*(BU(n-1))$ concentrated in even degrees d_{2n} has to be injective)

$$\Rightarrow d_{2n}(x e_{2n-1}) = c_n x$$

$x \in H^*(BU(n))$

get:

$$\text{ann}_{c_n} H^t BU(n) \xrightarrow{\cong} H^{t+2n-1} BU(n-1)$$

$$H^t BU(n-1) \xrightarrow{\cong} H^t BU(n) / c_n H^{t-2n} BU(n)$$

Inductively on t shows that

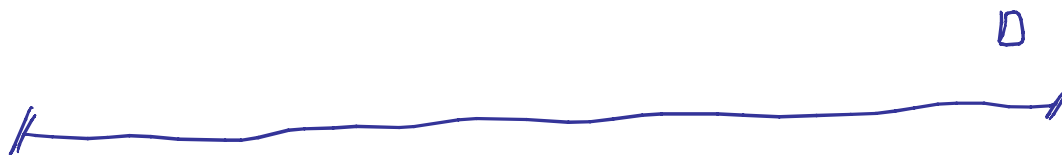
$H^t BU(n-1)$ concentrated in even degrees

$$\Rightarrow \begin{cases} H^t BU(n) \text{ concentrated in even degrees} \\ \text{ann } c_n = 0 \end{cases}$$

So we have

$$0 \rightarrow H^{t-2n} BU(n) \xrightarrow{c_n} H^t BU(n) \rightarrow H^t BU(n-1) \rightarrow 0$$

deduce $H^t BU(n) = \mathbb{Z}[c_1, \dots, c_n]$

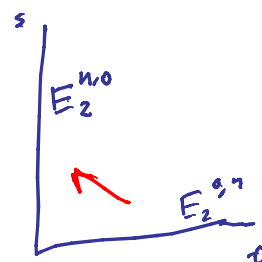


Aside: Edge Homomorphisms $F \xrightarrow{i} E \xrightarrow{p} B$

$$H^s(B; H^t(F)) \Rightarrow H^{st}(E)$$

$$\begin{array}{ccccc} H^n E & \leftarrow & E_\infty^{n,0} & \leftarrow & E_2^{n,0} \\ & & & & \parallel \\ & & & & H^n(B) \end{array}$$

$\swarrow \quad \searrow$
 P^*



$$\begin{array}{ccccc} H^n E & \rightarrow & E_\infty^{0,1} & \rightarrow & E_2^{0,1} \\ & & & & \parallel \\ & & & & H^n(F) \end{array}$$

\swarrow
 i^*

So we conclude:

$$BU(n-1) \hookrightarrow BU(n)$$

$$\mathcal{Z}(c_1, \dots, c_{n-1}) \longleftarrow \mathcal{Z}(c_1, \dots, c_n)$$

$$c_i \longleftarrow c_i$$

$$0 \longleftarrow c_n$$

Lemma Suppose $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ is classified by

$$X \xrightarrow{f} BU(n-1)$$

then the composite $X \rightarrow BU(n-1) \rightarrow BU(n)$

classifies $V \oplus \mathbb{R}$.

Pf:

P = principal $U(n-1)$ -bundle

$$V = P \times_{U(n-1)} \mathbb{C}^{n-1}$$

$$\begin{aligned} P \times_{U(n-1)} U(n) \times_{U(n)} \mathbb{C}^n &= P \times_{U(n-1)} \mathbb{C}^n \\ &= (P \times_{U(n-1)} \mathbb{C}^{n-1}) \times \mathbb{C} \quad \square \end{aligned}$$

Cor:

$$c_i(V \oplus R) = \begin{cases} c_i(V), & i < n \\ 0, & i = n \end{cases}$$

i.e. chern classes for rank $(n-1)$ bundles
compatible w/ chern classes of
rank n -bundles

\Rightarrow can define $c_i(V) \quad \forall i \in \mathbb{Z}$

$$(c_i(V) = 0 \quad i > \text{rk}(V))$$

proof of Cartan formula is deferred.....
(need more technology)

Splitting Principle

"Any formula involving chern classes
need only be checked on
sums of line bundles"