20 - Splitting principle + elementary symmetric poly's

Note Title 4/15/2010

Summary:

We computed $H^* BU(n) = \mathbb{Z}[c_1, \dots, c_n]$

$|c_i| = 2i$

$$\begin{array}{c} \begin{tikzcd} V \arrow[r] \arrow[d] & V^n_{univ} \arrow[d] \\ X \arrow[r, "f"'] & BU(n) \end{tikzcd} \end{array} \qquad \begin{array}{c} H^{2i}(X) \\ \cup\!\!\!\mid \\ c_i(V) := \begin{cases} f^* c_i & , 1 \le i \le n \\ 0 & , i > n \\ 1 & , i = 0 \end{cases} \end{array}$$

Satisfying:

Naturality: $c_i(g^* V) = g^* c_i(V)$

dim: $c_i(V) = 0$ if $i > \dim V$

Stability: $c_i(V) = c_i(V \oplus \mathbb{C})$

Normalization:
$$\begin{array}{c} L_{univ} \\ \downarrow \\ \mathbb{C}P^\infty \\ \wr \\ H^* = \mathbb{Z}[x] \end{array} \qquad c_1(L_{univ}) = x$$

$U(n-1) \hookrightarrow U(n)$

$BU(n-1) \longrightarrow BU(n)$

$i < n \qquad c_i \longleftarrow c_i$

$0 \longleftarrow c_n$

Goal: Cartan formula:

$$c_n(V \oplus W) = \sum_{n_1 + n_2 = n}{}' c_{n_1}(V)\, c_{n_2}(W)$$

Thm: (Splitting principle)

V = rk n cx V.B. $/X$

$\exists \quad \tilde{X} \xrightarrow[f]{} X$ s.t.

(1) $f^*V \cong L_1 \oplus \cdots \oplus L_n \qquad \mathrm{rk}(L_i) = 1$

(2) $f^* : H^*(X) \longrightarrow H^*(\tilde{X})$ is injective

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(pf) Suffices to prove you can split off one line.

$$\mathbb{C}P^{n-1} \longrightarrow P(V) \xrightarrow[g]{} X$$

projective space bundle

$$P(V) = \{ (x, L_x) \mid x \in X,\ L_x \subseteq V_x \text{ line} \}$$

$$\begin{array}{l} g^*V \underset{\text{subbundle}}{\supset} \{ (x, L_x, v) \mid v \in L_x \} =: L \\ \downarrow \\ P(V) \end{array}$$

Give g^*V a hermitian structure (can do this see BL(?))

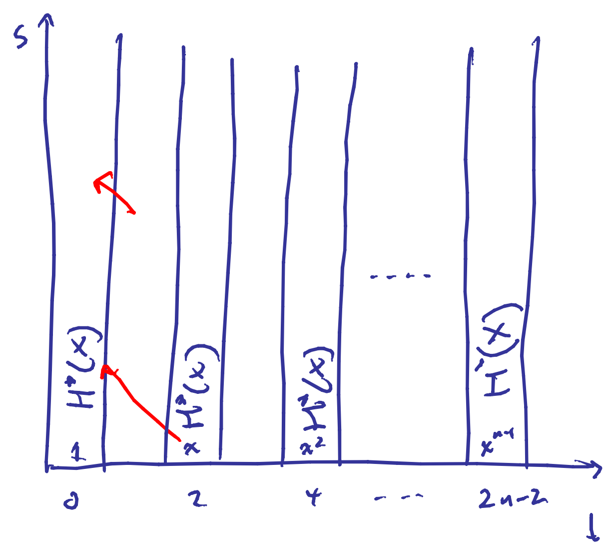
$\Rightarrow \quad g^*V \cong L \oplus L^\perp$

Just need to show:

$$g^* : H^*(X) \longrightarrow H^*(P(V))$$

is <u>injective</u>.

<u>SSS</u> $\quad H^s(X; H^t(\mathbb{C}P^{n-1})) \Rightarrow H^{s+t}(P(V))$



$$H^*(\mathbb{C}P^{n-1}) = \frac{\mathbb{Z}[x]}{(x^n)}$$

<u>Edge</u> <u>Homomorphism</u> $\Rightarrow$ need to argue no diff'ls

(suffices to show x is a P.C.)

Room for d_3 but...

Consider universal example:

$$\mathbb{C}P^{n-1} \longrightarrow P(V_{univ}) \xrightarrow{g_{univ}} BU(n)$$

SSS for g_{univ} concentrated in

$E_2^{s,t}(g_{univ})$ s, t even

$\Longrightarrow$ no diff'ls

$\Longrightarrow$ x_{univ} P.C.

$$\begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r] \arrow[d] & \mathbb{C}P^{n-1} \arrow[d] \\ P(V) \arrow[r] \arrow[d] & P(V_{univ}) \arrow[d] \\ X \arrow[r] & BU(n) \end{tikzcd} \Longrightarrow \begin{array}{c} E_2^{s,t}(g_{univ}) \longrightarrow E_2^{s,t}(g) \\ x_{univ} \longmapsto x \end{array}$$

$\Longrightarrow$ x P.C.

$\square$

Splitting principle $\Longrightarrow$ just need to prove

$$c_n(V \oplus W) = \sum_{n_1+n_2=n} c_{n_1}(V)\, c_{n_2}(W)$$

for $V, W =$ sums of line bundles

Conceptual Aside: Elementary symmetric polynomials & the maximal torus.

$$
\begin{tikzcd}
L_1 \oplus \cdots \oplus L_n \arrow[r] \arrow[d] & \cdots \arrow[r] & L_1 \oplus L_2 \oplus V^{n-2} \arrow[r] \arrow[d] & L_1 \oplus V^{n-1} \arrow[r] \arrow[d] & V^n_{univ} \arrow[d] \\
P(V_2) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & P(V^{n-1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & P(V^n_{univ}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & BU(n) \\
B\underbrace{U(1) \times \cdots \times U(1)}_{n} \arrow[r] & \cdots \arrow[r] & B(U(1) \times U(1) \times U(n-2)) & B(U(1) \times U(n-1)) &
\end{tikzcd}
$$

$\Big\}$ — class

i.e. the universal instance of the splitting principle is pullback on

$$B\,U(1)^n \longrightarrow BU(n)$$

$$\underset{\substack{\text{maximal} \\ \text{torus}}}{U(1)^n \hookrightarrow} U(n)$$

Lem $B(G \times H) \simeq BG \times BH$

(pf) principal $G \times H$ bundles are products of princ. G bundles & princ. H-bundles

$\square$

The claim follows from lemma:

Lem: $P(V^n_{univ}) \longrightarrow BU(n)$

is equivalent to $B(U(1) \times U(n-1)) \to BU(n)$

(pf)

$$\begin{array}{ccc} L_{univ} \boxplus V^{n-1}_{univ} & \longrightarrow & V^n_{univ} \\ \downarrow & & \downarrow \\ B(U(1) \times U(n-1)) & \longrightarrow & BU(n) \end{array}$$

external sum

$\frac{U(n)}{U(1) \times U(n-1)} \longrightarrow B(U(1) \times U(n-1))$

$\shortparallel$

$BU(1) \times BU(n-1)$

$U(n) \circlearrowright \mathbb{C}P^{n-1}$ transitively w/ stabilizer $U(1) \times U(n-1)$

$\Rightarrow$ fiber is $\mathbb{C}P^{n-1}$

$$B(U(1)\times U(n-1)) = EU(n)/_{U(1)\times U(n-1)}$$

$$= EU(n)\times_{U(n)} \underbrace{U(n)/_{U(1)\times U(n-1)}}_{\mathbb{C}P^{n-1}}$$

<u>But</u>

$$V^n_{\text{univ}} = EU(n)\times_{U(n)} \mathbb{C}^n$$

$$\Rightarrow P(V^n_{\text{univ}}) = EU(n)\times_{U(n)} P(\mathbb{C}^n)$$

$\square$

<u>Consequence</u>

$$\begin{tikzcd}
L_{\text{univ}}\boxplus\cdots\boxplus L_{\text{univ}} \arrow[r] \arrow[d] & V^n_{\text{univ}} \arrow[d] \\
BU(1)^n \arrow[r, "\phi"'] \arrow[ur, phantom, "\lrcorner"] & BU(n)
\end{tikzcd}$$

$$\mathbb{Z}[c_1,\dots,c_n] = H^* BU(n) \overset{\phi^*}{\hookrightarrow} H^* BU(1)^n = \mathbb{Z}[x_1,\dots,x_n]$$

Let NT^n = normalizer of torus $T^n \subset U(n)$

$$NT^n / T^n \cong \Sigma_n \quad \text{"Weyl gp"}$$

$\Sigma_n \circlearrowright T^n$ by conjugation, action permutes factors.

<u>HW:</u> Conjugation induces trivial action on H^*

$$\Rightarrow \quad H^*(BU(n)) \circlearrowleft NT^n \text{ acts trivially}$$

$$\Rightarrow \quad H^*(BU(n)) \hookrightarrow \left(H^* BU(1)^n\right)^{\Sigma_n}$$

$$\cong$$

$$\mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n}$$

symmetric polynomials

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

$$\sigma \in \Sigma_n$$

Classically:

$$\mathbb{Z}(x_1,\dots,x_n)^{\Sigma_n} = \mathbb{Z}[e_1,\dots,e_n]$$

$$e_1(x_1,\dots,x_n) = x_1 + x_2 + \dots + x_n$$
$$= \sum_{i_1} x_{i_1}$$

$$e_2(x_1,\dots,x_n) = \sum_{i_1<i_2} x_{i_1} x_{i_2}$$

$$\vdots$$

$$e_n(x_1,\dots,x_n) = x_1 \cdots x_n$$

Claim 1

$$\mathbb{Z}[c_1,\dots,c_n] \hookrightarrow \mathbb{Z}[e_1,\dots,e_n]$$

is an isomorphism

$$c_i \longmapsto e_i$$