21 - Cartan formula, Thom isomorphism thm

Note Title 4/15/2010

<u>Recall:</u>

$$\begin{array}{ccc} H^* BU(n) & \hookrightarrow & H^*(BU(1)^n)^{\Sigma_n} \\ \| & & \| \\ \mathbb{Z}[c_1, \dots, c_n] & & \mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n} \end{array}$$

<u>Classically:</u>

$$\mathbb{Z}(x_1, \dots, x_n)^{\Sigma_n} = \mathbb{Z}[e_1, \dots, e_n]$$

$$e_1(x_1, \dots, x_n) = x_1 + x_2 + \dots + x_n$$

$$= \sum_{i_1} x_{i_1}$$

$$e_2(x_1, \dots, x_n) = \sum_{i_1 < i_2}' x_{i_1} x_{i_2}$$

$$\vdots$$

$$e_n(x_1, \dots, x_n) = x_1 \cdots x_n$$

<u>Claim:</u> $\mathbb{Z}[c_1, \dots, c_n] \hookrightarrow \mathbb{Z}[e_1, \dots, e_n]$

is an isomorphism

$$c_i \longmapsto e_i$$

Claim 1 is equivalent to Cartan formula.

Assume cartan formula:

$$p_i : BU(1)^n \longrightarrow BU(1) \quad i^{\text{th}} \text{ proj.}$$

$$L_i = p_i^* L_{\text{univ}}$$

$$\Rightarrow \quad L_{\text{univ}} \boxplus \cdots \boxplus L_{\text{univ}} = L_1 \oplus \cdots \oplus L_n$$

$$\phi^*(c_i) = c_i(L_{\text{univ}} \boxplus \cdots \boxplus L_{\text{univ}})$$

$$= c_i(L_1 \oplus \cdots \oplus L_n)$$

$$= \sum_{i_1 + \cdots + i_n = i} c_{i_1}(L_1) \cdots c_{i_n}(L_n)$$

induction cartan

only set contributions from

$$c_{j_1} = \cdots = c_{j_i} = 1 \quad 0 \leq j_1 < j_2 < \cdots < j_i \leq n$$

all others 0

$$= \sum_{0 \le j_1 < j_2 < \cdots < j_i \le n} x_{j_1} \cdots x_{j_n} = e_i(x_1, \ldots, x_n)$$

Conversely: Cartan formula is verified if it is verified for the universal example of a sum of line bundles (splitting princ.)

$$c_i(L_{univ} \boxplus \cdots \boxplus L_{univ}) = \phi^*(c_i) = e_i$$

verifies Cartan in this case.

(pf of Cartan formula up to a constant)

Know: in $H^* BU(1)^m = \mathbb{Z}[x_1, \ldots, x_n]$

$$\begin{array}{c} L_i \\ \downarrow \\ BU(1)^m \end{array} = p_i^* L_{univ}$$

$$c_1(L_i) = x_i$$

$$c_n\left(\bigoplus_{i=1}^m L_i\right) = P_{n,m}(x_1, \ldots, x_m)$$

Since $BU(1)^m$ carries universal example of a sum of m line bundles

$$\Rightarrow \quad c_n\left(\bigoplus_{i=1}^{m} L_i\right) = p_{n,m}\left(c_1(L_1), \ldots, c_1(L_m)\right)$$

for any collection $\{L_i\}$ of line bundles over any space.

(1) $p_{n,m}$ is symmetric

$$\left(\text{since} \quad \bigoplus_{i=1}^{m} L_i \cong \bigoplus_{i=1}^{m} L_{\sigma(i)}\right)$$

$$\sigma \in \Sigma_m$$

$$\Rightarrow \quad p_{n,m}(x_1, \ldots, x_m) \in \mathbb{Z}[e_1, \ldots, e_m] \subset \mathbb{Z}[x_1, \ldots, x_m]$$

(2) <u>$n = m$</u>: $\quad L_n = \mathbb{C}$

$$c_n\left(\bigoplus_{i=1}^{n} L_i\right) \underset{\text{stability}}{=} c_n\left(\sum_{i=1}^{n-1} L_i\right) \underset{\dim}{=} 0$$

$$\Rightarrow \quad p_{n,n}(x_1, \ldots, x_n) = 0$$

$$\Rightarrow \quad p_{n,n} = a_n e_n$$

$$e_n(x_1, \ldots, x_n) = x_1 \cdots x_n$$

$$a_n \in \mathbb{Z}$$

(3) $\quad P_{n,m}(x_1, \dots, x_n, 0, \dots, 0) \underset{\text{stability}}{=} P_{n,n}(x_1, \dots, x_n)$

$$P_{n,n}(x_1, \dots, x_n) = a_n e_n$$

$\Longrightarrow$ (writing $P_{n,m}$ in terms of elementary sym)

$$P_{n,m}(x_1, \dots, x_m) = a_n e_n(x_1, \dots, x_m)$$

$$= a_n \sum_{0 \le i_1 < \dots < i_n \le m}' x_{i_1} \cdots x_{i_n}$$

<u>So:</u> $\quad c_n(L_1 \oplus \cdots \oplus L_m) = a_n\, e_n(c_1(L_1), \dots, c_1(L_m))$, $\square$

$\left[\text{Remains to show} \quad a_n = 1\right]$

Thom Isomorphism & euler classes

Note Title 4/22/2010

Thom Complex "twisted suspension"

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \quad \text{real or cx v.b. (give it a metric)}$$

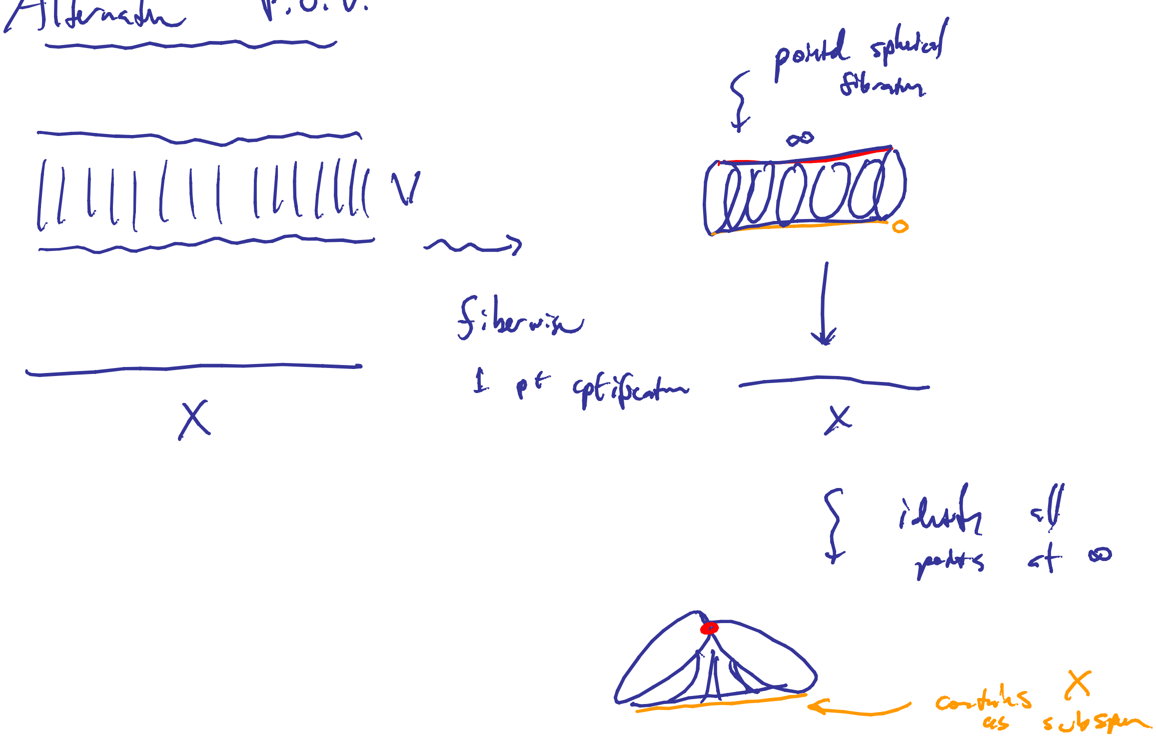
$S(V)$ = associated unit sphere bundle

$D(V)$ = associated unit disk bundle

Def' the Thom complex of $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ is

$$X^V := D(V)/S(V)$$

Alternate P.O.V.



e.g. V = trivial bundle

$$X^{\mathbb{R}^n} = \frac{X \times D^n}{X \times S^{n-1}} = X_+ \wedge S^n = \Sigma^n X_+$$

Orientation theory R = ring, $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ v.b. $\dim_{\mathbb{R}} V = n$

fibration pair:

$$(D^n, S^{n-1}) \longrightarrow (D(V), S(V)) \longrightarrow X$$

$\Longrightarrow$ $H^*(D^n, S^{n-1}; R)$ is a local coeff system.

(i.e. has π_1 action)

Def: V is R-orientable if

$$\pi_1 \circlearrowright \underset{\underset{R}{\parallel}}{H^n(D^n, S^{n-1}; R)} \text{ trivial action}$$

(This is equivalent to whatever you learned in 905)

Rmk: (univ. coef)

$$\underset{\underset{\pi_1}{\circlearrowright}}{H^n(D^n, S^{n-1}; R)} \cong \underset{\underset{\pi_1}{\circlearrowright}}{H^n(D^n, S^{n-1})} \otimes R$$

Thus: $\mathbb{Z}$-orientable $\Rightarrow$ R-orientable

Rmk: $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2$

$\Rightarrow$ if π_1 acts non-trivially,

$\exists$ 2-fold cover: $p: X' \to X$

$p^* V$ $\mathbb{Z}$-orientable $\Rightarrow$ R orientable

[every cx v.b. is $\mathbb{Z}$ orientable]

Rmk: $\mathrm{Aut}(\mathbb{Z}/2) = 1$

$\Rightarrow$ every v.b. is $\mathbb{Z}/2$ orientable.

Thom Isomorphism Thm:

Suppose V is R-orientable

$\Rightarrow$ there is an iso

$$\tilde{H}^{*+n}(X^V; R) \cong H^*(X; R)$$

Rmk: $\tilde{H}^{*+n}(\Sigma^{\infty} \Sigma^n X_+; R) \cong \tilde{H}^*(X_+; R) \cong H^*(X; R)$

So: Thom iso says:

"X^V looks like $X^{\mathbb{R}^n} \approx \Sigma^n X_+$

to the eyes of $\tilde{H}^*(-; R)$"

Pf of Thom iso:

SSS

$$H^s(X; H^t(D^n, S^{n-1}; R)) \Rightarrow H^{s+t}(D^n(V), S^{n-1}(V); R)$$

$\| \wr \leftarrow$ R-orientability $\qquad\qquad \| \;\; \tilde{H}^{s+t}(X^V; R)$

$$\begin{cases} H^s(X; R) & ; \; t = n \\ 0 & ; \; o/w \end{cases}$$

depends on choice of generator in $H^n(D^n, S^{n-1}; R)$

$\updownarrow$

choice of R-orientation

SSS collapses to Thom iso:

$$H^s(X; R) \cong \tilde{H}^{s+n}(X^V; R)$$

$\square$