22 - Thom + Euler classes

Note Title 4/22/2010

Thom iso: $H^0(X;R) \cong H^n(X^V;R)$

$$\begin{array}{ccc} H^0(X;R) & \cong & H^n(X^V;R) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1 & \longmapsto & [V] \end{array}$$

Thom class

Functoriality of thom cx:

$$\begin{tikzcd} f^*V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \Rightarrow f_! : X^{f^*V} \to Y^V$$

Homework: $X \times Y^{V \boxplus W} \simeq X^V \wedge Y^W$

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \boxplus V \arrow[d] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

gives rise to

$$X^V \xrightarrow{\Delta_!} (X \times X)^{0 \boxplus V} \simeq X^0 \wedge X^V$$
$$\simeq X_+ \wedge X^V$$

"Thom diagonal"

on cohomology

$$H^*(X;R) \otimes \tilde{H}^*(X^V;R) \longrightarrow \tilde{H}^*(X^V;R)$$

$\Longrightarrow$ $\tilde{H}^*(X^V;R)$ is a $H^*(X;R)$-module

Thom iso $\Longrightarrow$ $\tilde{H}^*(X^V;R)$ is free of rank 1 as a $H^*(X;R)$-module

(generator = Thom class)

$$\begin{pmatrix}\text{Choice} \\ \text{of Thom class}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}R\text{-orientation}\end{pmatrix}$$

X connected

$$\{R\text{-orientations}\} \cong R^\times$$

Lemma: $[V] \in \tilde{H}^n(X^V)$ Thom classes

$[W] \in \tilde{H}^m(Y^W)$

$$[V]\cdot[W] \in \tilde{H}^{n+m}(X^V \wedge Y^W) = \tilde{H}^{n+m}(X \times Y^{V \boxplus W})$$

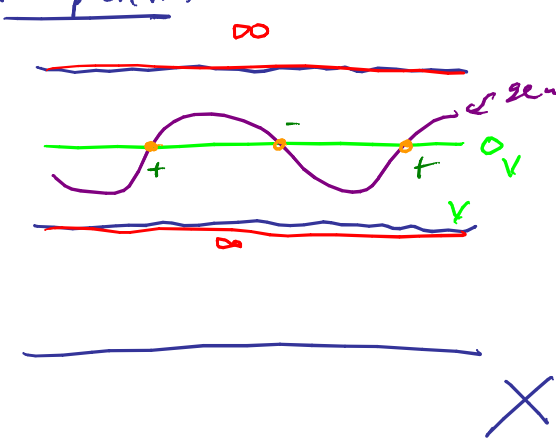
Euler class:

$$X_+ \hookrightarrow X^V \qquad \text{zero section}$$
$$x \longmapsto 0_x$$
$$+ \longmapsto \infty$$

$$\tilde{H}^n(X^V) \longrightarrow H^n(X)$$
$$[V] \longmapsto e(V) \qquad \text{Euler class}$$

V = v.b. $\dim \mathbb{R} = n$

Geometric interpretation: X = closed orientable mfld dim d



$s \cap 0_V$

= oriented codim n submanifold of X

$\Big\}$ PD

elt of $\tilde{H}^n(X)$

$(e(V))$

Thm: M connected, oriented

$$e(TM) \in H^d(M)$$
$$\shortparallel$$
$$\chi(M)[M]$$

$\Big(e(TM) = I(s)[M]$ — # zeros of s counted w/ multiplicity — Poincaré-Hopf thm. $\Big)$

Lem: $e(V \oplus W) = e(V)\, e(W)$

(pf)

$$\begin{tikzcd}
V \oplus W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \boxplus W \arrow[d] \\
X \arrow[r, "\Delta"'] & X \times X
\end{tikzcd}$$

$$\begin{tikzcd}
(X \times X)^{V \boxplus W} \arrow[r, phantom, "\simeq"] & X^V \wedge X^W & X^{V \oplus W} \arrow[l] \\
X \times X_+ \arrow[u, hook] \arrow[r, phantom, "\simeq"] & X_+ \wedge X_+ \arrow[u, hook] & X_+ \arrow[l, "\Delta"] \arrow[u, hook]
\end{tikzcd}$$

$$[V \boxplus W] \mapsto [V] \otimes [W] \leftrightarrow [V \oplus W]$$

$$[V] \otimes [W] \mapsto e(V) \otimes e(W) \qquad [V \oplus W] \mapsto e(V \oplus W)$$

$$e(V) \otimes e(W) \mapsto e(V)\, e(W)$$

Euler classes of Chern classes

Note: V = cx v.b. $\Rightarrow$ V has canonical $\mathbb{Z}$-orientation.

Thm

V = cx v.b.$/X$, $\dim_{\mathbb{C}} V = n$

$$c_n(V) = e(V) \in H^{2n}(X)$$

(b) Suffices to prove this in universal case.

$BU(n)^{V^n_{univ}}$??

$$\begin{array}{c} V^n_{univ} = EU(n) \times_{U(n)} \mathbb{C}^n \\ \downarrow \\ BU(n) \end{array}$$

$$S(V^n_{univ}) = EU(n) \times_{U(n)} S^{2n-1}$$

$$= EU(n) \times_{U(n)} U(n)/U(n-1)$$

$$= EU(n)/U(n-1)$$

$$= BU(n-1)$$

$$\begin{tikzcd} S(V^n_{univ}) \arrow[r] \arrow[d, "\wr"] & D(V^n_{univ}) \arrow[d, "\simeq"] \\ BU(n-1) \arrow[r] & BU(n) \end{tikzcd}$$

$$\Longrightarrow \quad BU(n)^{V^n_{univ}} \simeq \mathrm{Cofiber}\left(BU(n-1) \hookrightarrow BU(n) \right)$$

$$H^{2n}(BU(n-1)) \xleftarrow[\iota^*]{} H^{2n}(BU(n)) \longleftarrow \tilde{H}^{2n}\left(BU(n)^{V^n_{univ}}\right) \longleftarrow 0$$

can take this to be induced by zero section

$$\tilde{H}^{2n}\left(BU(n)^{V^n_{univ}}\right) \cong \ker(\iota^*) = \mathbb{Z}\{c_n\}$$

$$\begin{array}{ccc} \psi & & \psi \\ [V^n_{univ}] & \longmapsto & c_n \end{array}$$

$$\Longrightarrow \quad c_n = e(V^n_{univ})$$

(actually $c_n = \pm e$ see below)

$\square$

<u>Sum formula</u>: $L_1, \ldots, L_n$ cx v.b.'s $/X$

reduced to constant:

$$c_n(L_1 \oplus \cdots \oplus L_n) = a_n\, c_1(L_1) \cdots c_1(L_n)$$

$$a_n \in \mathbb{Z}$$

(want $a_n = 1$)

$$c_n(L_1 \oplus \cdots \oplus L_n) = e(L_1 \oplus \cdots \oplus L_n)$$

$$= e(L_1) \cdots e(L_n)$$

$$= c_1(L_1) \cdots c_1(L_n)$$

$\Rightarrow$ gives Cartan formula!

Rmk on signs

c_n only defined modulo sign

Fix that sign so that

$c_n = e$