23 - Steenrod operations - Stiefel - Whitney Classes

Note Title 4/29/2010

Axiomatics

We will prove

There are natural homomorphisms!

$$Sq^i : H^*(- ; \mathbb{F}_2) \longrightarrow H^{*+i}(- ; \mathbb{F}_2)$$

(and on reduced coh)

Satisfying

(1) $|x| = n$

$$Sq^0(x) = x$$

$$Sq^n(x) = x^2$$

$$Sq^i(x) = 0 \qquad i > n$$

$$\sigma : \tilde{H}^*(X) \cong \tilde{H}^{*+1}(\Sigma X)$$

(2) $$Sq^i(\sigma x) = \sigma\, Sq^i(x)$$

(3) $$Sq^i(x \cup y) = \sum_{i_1 + i_2 = i} Sq^{i_1}(x) \cup Sq^{i_2}(y)$$

We will then define characteristic classes

$$\left(\begin{array}{cl} V & \text{rank } n \\ \downarrow & \text{real v.b.} \\ X & \end{array}\right) \rightsquigarrow w_i(V) \in H^i(X; \mathbb{F}_2)$$

$$\begin{array}{ccc} \tilde{H}^{n+i}(X^V) & \xleftarrow{\cong} & H^i(X) \\ \Psi & & \Psi \\ Sq^i[V] & \longmapsto & w_i(V) \end{array}$$

s.t. $w_0(V) = 1$

(1) $w_i(V) = 0 \qquad i > \dim_{\mathbb{R}} V$

(2) $w_i(V \oplus \mathbb{R}) = w_i(V)$

(3) L_{univ} $\downarrow$ $\mathbb{R}P^\infty$ univ line bundle

$w_1(L_{univ}) = x$

$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$

$|x| = 1$

(4) $$w_i(V \oplus W) = \sum_{i_1 + i_2 = i} w_{i_1}(V)\, w_{i_2}(W)$$

Universal case:

$$H^* BO(n) = \mathbb{F}_2[w_1, w_2, \ldots, w_n]$$

$$O(1)^n \xhookrightarrow[\text{diagonal}]{} O(n)$$

gives inclusion (splitting principle)

$$\begin{array}{ccc} H^*(BO(n); \mathbb{F}_2) & \xhookrightarrow[\cong]{} & H^*(BO(1)^n; \mathbb{F}_2)^{\Sigma_n} \\ \| & & \| \\ \mathbb{F}_2[w_1, \ldots, w_n] & & \mathbb{F}_2[x_1, \ldots, x_n]^{\Sigma_n} \end{array}$$

Note $BO(1) = \mathbb{R}P^\infty$

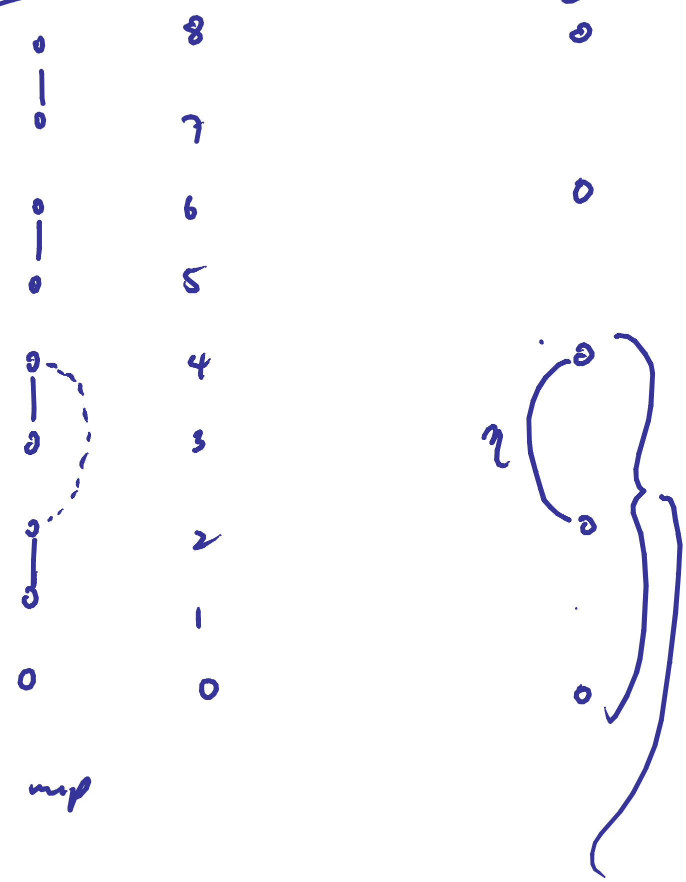
$|x_i| = 1$

$$w_i \longmapsto e_i(x_1, \ldots, x_n)$$

Problem: $X = $ CW cx

want to use H^* to "see" the attaching maps that bind the cells of X

e.g. $\mathbb{R}P^\infty = B\mathbb{Z}/2 \xrightarrow{\mathbb{Z}/2 \hookrightarrow S^1} \mathbb{C}P^\infty = BS^1$



$| = \cdot 2$ map

$\mathbb{Z}/2$'s in $H^{2i}(\mathbb{R}P^\infty)$ detect the presence of $\cdot 2$ attaching map

$\mathbb{C}P^2 = C\eta$

$\eta : S^3 \to S^2$

but η is "invisible" to H^*! (or is it? Q!)

Hopf invariant $\qquad \alpha : S^3 \longrightarrow S^2$

$$\alpha \left\{ \begin{matrix} 0 & 4 & y \\ 0 & 2 & x \end{matrix} \right.$$

$$x \in H^2(C_\alpha)$$

$$x^2 = H.I.(\alpha) \cdot y \in H^4(C_\alpha)$$

So the squaring operation

$$H^2(-) \longrightarrow H^4(-)$$

"detects α"

Natural transformation

$\updownarrow$ Yoneda

$$[K(\mathbb{Z},2), K(\mathbb{Z},4)] = H^4(K(\mathbb{Z},2), \mathbb{Z}) \ni (\iota_2)^2$$

fund. class

Natural transformations

$$H^n(-;A) \longrightarrow H^{n+k}(-;B)$$

"cohomology operations"

$\updownarrow$

$H^{n+k}(K(A,n);B)$ "characteristic classes of cohomology classes"

Idea: Cohomology operations "detect" attaching maps (thought of as lying in $\pi_*(S^n)$)

Problem: $H^*(K(A,n);B)$

$H^*(K(\mathbb{Z},n);\mathbb{Z})$ Really messy: p-torsion for every prime, ...

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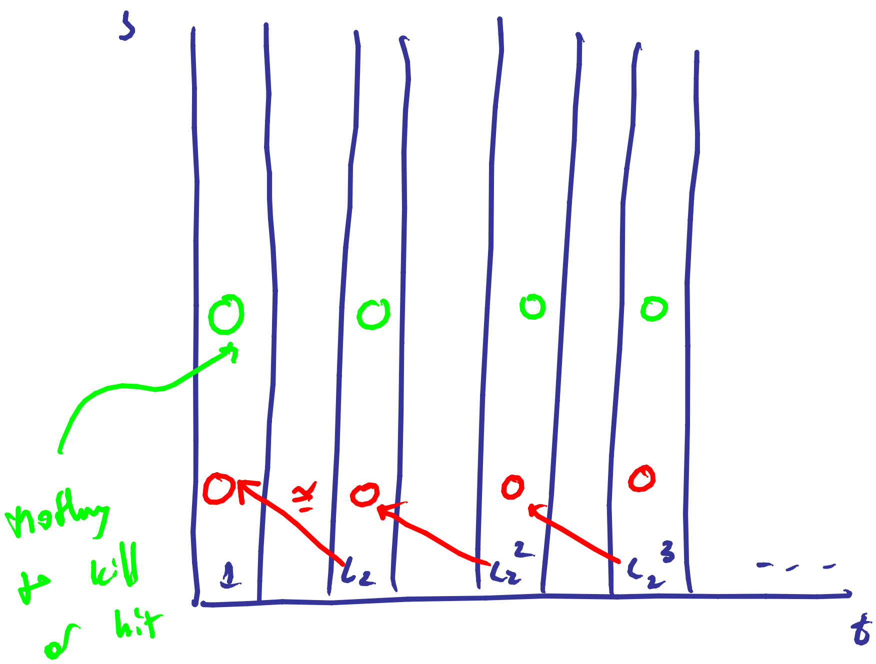
e.g. $H^*(K(\mathbb{Z},n);\mathbb{Q})$

$n=1$: $H^*(S^1;\mathbb{Q}) = \Lambda_{\mathbb{Q}}[c_1]$

$n=2$: $H^*(\mathbb{C}P^\infty;\mathbb{Q}) = \mathbb{Q}[c_2]$

$K(\mathbb{Z},2) \to * \longrightarrow K(\mathbb{Z},3)$

$$H^s(K(\mathbb{Z},3);\, H^t(K(\mathbb{Z},2);\mathbb{Q})) \Rightarrow H^{s+t}(*;\mathbb{Q})$$



$$d_3\iota_2^n = n\,\iota_2^{n-1} d_3\iota_2$$

iso since working / $\mathbb{Q}$

Can't have anything more

$$\Rightarrow \quad H^*(K(\mathbb{Z},3);\mathbb{Q}) = \Lambda_{\mathbb{Q}}[\iota_3]$$

Continue:

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) = \begin{cases} \Lambda_{\mathbb{Q}}[\iota_n], & n \text{ odd} \\ \underline{\Lambda}_{\mathbb{Q}}[\iota_n], & n \text{ even} \end{cases}$$

$\Longrightarrow$ Rational cohomology operations (n,k)

n odd $\Longrightarrow$ none
$k > n$

n even
$k = in$ $\Longrightarrow$ $x \longmapsto x^i$

$$S^{in-1} \xrightarrow[\alpha]{} S^n \longrightarrow C_\alpha$$

$$C_\alpha \quad \begin{array}{lll} 0 & in & y \\ \alpha & & \\ 0 & n & x \end{array}$$

$x^i = c \cdot y$

<u>problem:</u> $i > 2$
$x^2 = 0 \Rightarrow x^i = 0$

So in fact only $x \mapsto x^2$ is the only rational cohomology operation based detected on $\pi_*(S^{2n})$
(HI)

Serre used this to prove:

Thm: the only homotopy gps of spheres which do not consist entirely of torsion are

$$\pi_{4n-1}(S^{2n})$$

Similarly p-torsion is picked up by mod p cohomology operations.

$$H^*(K(\mathbb{F}_p, n); \mathbb{F}_p)$$

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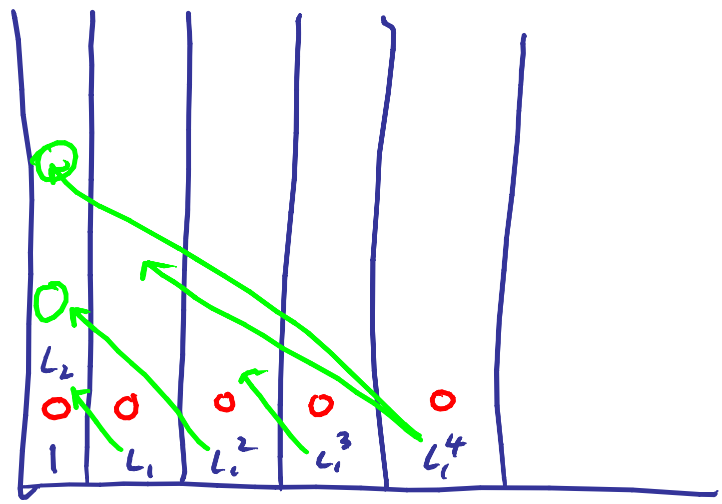
p = 2:

$n=1$ $\mathbb{R}P^\infty$: $\quad H^*(K(\mathbb{F}_2,1);\mathbb{F}_2) = \mathbb{F}_2[\iota_1]$

$$K(\mathbb{F}_2,1) \longrightarrow * \longrightarrow K(\mathbb{F}_2,2)$$

$$H^s(K(\mathbb{F}_2,2); H^t(K(\mathbb{F}_2,1);\mathbb{F}_2)) \Rightarrow H^{s+t}(*;\mathbb{F}_2)$$

<u>SSS</u>



$$d_2(\iota_1) = \iota_2$$

$$d_2(1 \otimes \iota_1^n) = n\,\iota_1^{n-1}\iota_2 = \begin{cases} 0 & n \text{ even} \\ \iota_1^{n-1}\iota_2 & n \text{ odd} \end{cases}$$

$$d_3(\iota_1^2) =: Sq^1(\iota_2) \in H^3(K(\mathbb{F}_2,2);\mathbb{F}_2)$$

$$d_3(\iota_1^{2n}) = \iota_1^{2n-2}\, Sq^1(\iota_2)$$
n odd

$$d_4(\iota_1^4) = 0 \qquad \text{(no targets)}$$

$$d_5(\iota_1^4) =: Sq^2 Sq^1(\iota_2)$$
$$\in H^5(K(\mathbb{F}_2,2), \mathbb{F}_2)$$

$\vdots$

<u>Find:</u> $$d_{2^n+1}(\iota_1^{2^n}) =: Sq^{2^{n-1}} \cdots\cdots Sq^1(\iota_2)$$

(determines all diff'ls)

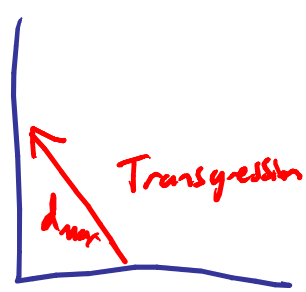
<u>Consequence:</u>

(1) $$H^*(K(\mathbb{F}_2,2)) = \mathbb{F}_2[Sq^{2^{n-1}}Sq^{2^{n-2}}\cdots Sq^1(\iota_2) \,;\, n \geq 0]$$

(2) we get cohomology operations

$$H^2(-;\mathbb{F}_2) \longrightarrow H^{2+1+2+\cdots+2^{n-1}}(-;\mathbb{F}_2)$$
$$x \longmapsto Sq^{2^{n-1}}Sq^{2^{n-2}}\cdots\cdots Sq^1(x)$$

(3)



$$d_{max}(\iota_1^{2^n}) = Sq^{2^{n-1}}\cdots\cdots Sq^1(\iota_2)$$

<u>Means</u>

$$\begin{array}{ccc} \tilde{H}^1(X;\mathbb{F}_2) & \xrightarrow{\cong} & \tilde{H}^2(\Sigma X;\mathbb{F}_2) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & \sigma x \end{array}$$

$$\underbrace{Sq^{2^{n-1}} Sq^{2^{n-2}} \cdots Sq^{1}}_{n} (\sigma x) = \sigma\left(x^{2^n}\right)$$

"i.e. Sq is a squaring operation applied n times"

<u>Moral:</u> even though ΣX has no cup products, squaring operations PERSIST mod 2.

$x \longmapsto x^{2^n}$ homomorphism (mod 2)

<u>Fact:</u> $Sq^{2^{n-1}} \cdots\cdots Sq^{1}$ is a homomorphism mod 2

$\mathbb{RP}^2_+$ mod 2

$$\begin{array}{ccc} 2 & 0 & x^2 \\ 1 & 0 & x \\ 0 & 0 & \end{array}$$

Squaring operation is detecting mult by 2 attaching map

$\Sigma \mathbb{RP}^2_+$ Sq^1

$$\begin{array}{cc} 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{array}$$

Sq^1 detects mult by 2