24 - Steenrod + S-W: cont'd

Note Title 4/29/2010

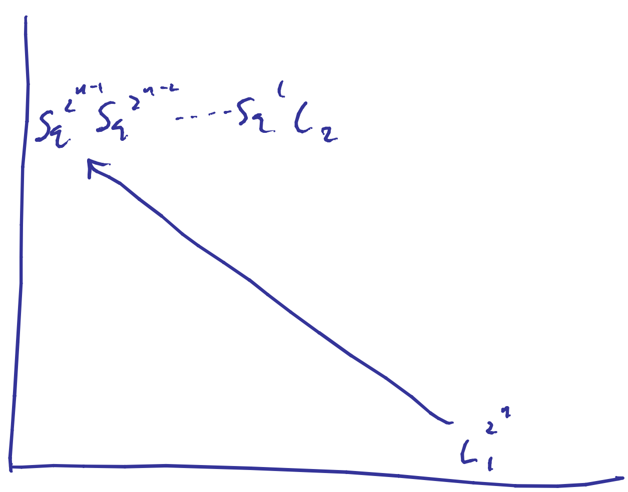
Recall: $H^{n+i}(K(\mathbb{F}_p, n); \mathbb{F}_p) \leftrightsquigarrow$ coh ops $H^n(-;\mathbb{F}_p) \to H^{n+i}(-;\mathbb{F}_p)$

$(p=2)$ (all coh w/ mod 2 coef)

$$H^*(K(\mathbb{F}_2, 1)) = \mathbb{F}_2[\iota_1]$$

$$H^*(K(\mathbb{F}_2, 2)) = \mathbb{F}_2[Sq^{2^{n-1}} Sq^{2^{n-2}} \cdots Sq^1 \iota_2]$$

$$K(\mathbb{F}_2, 1) \to * \to K(\mathbb{F}_2, 2)$$



$$\Sigma \Omega X \to X$$

$$\begin{tikzcd} \tilde{H}^i(X) \arrow[r, "\sigma"] & \tilde{H}^i(\Sigma \Omega X) \cong \tilde{H}^{i-1}(\Omega X) \arrow[l, bend left, dashed, "d_i"] \end{tikzcd}$$

lemma: $d_i = \sigma^{-1}$

$$Sq^{2^{n-1}} \cdots Sq^1(\sigma x) = x^{2^n}$$

In particular have $Sq^1 : H^2 \longrightarrow H^3$

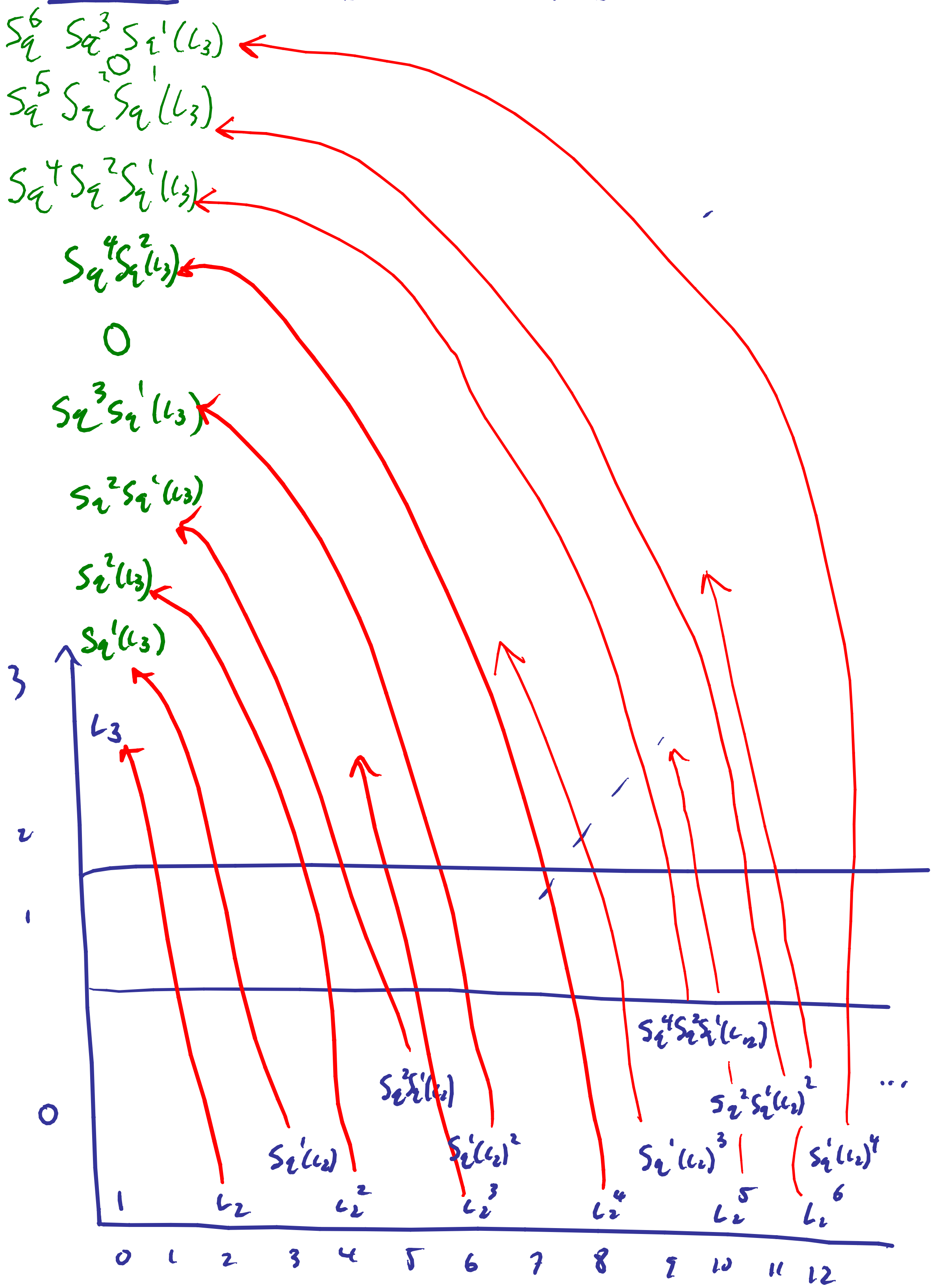
$\underline{\underline{\mathbb{RP}^2}}$:
$$\begin{array}{ccc} 2 & \circ & x^2 \\ & \big| \cdot 2 & \\ 1 & \circ & x \end{array}$$

$\underline{\underline{\Sigma \mathbb{RP}^2}}$:
$$\begin{array}{ccc} 3 & \circ & Sq^1(y) \\ & \big| \cdot 2 & \\ 2 & \circ & \sigma(x) = y \end{array}$$

"Sq^1 sees $\cdot 2$ attaching map"

(even though it is invisible to $H^*(-;\mathbb{F}_2)$)

Onwards (sketch) $K(\mathbb{F}_2, 3)$



Consequences:

(1) $H^*(K(\mathbb{F}_2,3);\mathbb{F}_2) \cong \mathbb{F}_2[Sq^I \iota_3]_I$

$I = (i_1, i_2, \ldots\ldots i_e)$

Satisfies: $\quad i_j \geq 2 i_{j+1}$ "admissible"

$i_j < i_{j+1} + i_{j+2} + \cdots + i_e + 3$

"inexcessive"

(2) We get cohomology operations

$Sq^I : H^3(-;\mathbb{F}_2) \to H^{3+\sum i_j}(-;\mathbb{F}_2)$

(In particular Sq^1 and Sq^2)

(3) $Sq\, \sigma x = \sigma\, Sq^I x$

$|x| = 2$

$$Sq^{2^n+1}\left(\sigma Sq^{2^{n-1}} \cdots\cdots Sq^1 x\right)$$

$$\parallel$$

$$\sigma\left[\left(Sq^{2^{n-1}} \cdots\cdots Sq^1 x\right)^2\right]$$

In general!

$$H^*(K(\mathbb{F}_2,n);\mathbb{F}_2) = \mathbb{F}_2[Sq^I \iota_n]$$

$$I = (i_1, \dots, i_\ell)$$

admissible $\quad i_j \geq 2 i_{j+1}$

inexcessive $\quad i_j < i_{j+1} + \cdots + i_\ell + n$

In particular

get: $Sq^i x \qquad 1 \leq i < |x|$

Convention: $Sq^0(x) = x$

$Sq^{|x|}(x) = x^2$

$Sq^i(x) = 0 \quad i > 0$

With this convention (using transgressions in SSS)

$Sq^i(\sigma x) = \sigma Sq^i(x)$

($\Longrightarrow$ Sq^i are homomorphisms)

$[\Sigma_i X, K(\pi, n)]_*$

pinch map gives addition

Cartan formula: omitted... (Hatcher)

[typically Sq^i are first constructed via a "mysterious construction" satisfying Cartan]

Stiefel-Whitney Classes

will omit $\int$

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \rightsquigarrow w_i(V) \in H^i(X;\mathbb{F}_2)$$

$\dim_{\mathbb{R}} V = n$

$$\begin{array}{ccc} \tilde{H}^{i+n}(X^V) & \overset{\Phi}{\underset{\text{Thom}}{\xleftarrow{\cong}}} & H^i(X) \\ \Psi & & \Psi \\ Sq^i[V] & \longmapsto & w_i(V) \end{array}$$

AKA: $Sq^i[V] = w_i(V)[V]$

Properties:

Naturality: follows from naturality of Sq^i and Thom class

Dimension: $Sq^i[V] = 0 \quad i > n$

$$\Rightarrow w_i(V) = 0 \quad i > n$$

$$Sq^0([V]) = [V] = \Phi(1)$$

$$\Rightarrow w_0(V) = 1$$

Stability: $X^{V\oplus\mathbb{R}} = \Sigma X^V$

$$[V\oplus\mathbb{R}] = \sigma[V]$$

$$Sq^i[V\oplus\mathbb{R}] = Sq^i\sigma[V] = \sigma Sq^i[V] = w_i(V)\sigma[V]$$

$$w_i(V)[V\oplus\mathbb{R}] \;\;''$$

$$\Rightarrow \quad w_i(V \oplus \mathbb{R}) = w_i(V)$$

Sum formula:

$$X^{V \oplus W} = X^V \wedge X^W$$

$$[V \oplus W] \mapsto [V][W]$$

$$Sq^i([V][W]) = \sum_{i_1+i_2=i} Sq^{i_1}([V])\, Sq^{i_2}([W])$$

$$= \sum_{i_1+i_2=i} w_{i_1}(V)\, w_{i_2}(W)\, [V][W]$$

$$\Rightarrow \quad w_i(V \oplus W) = \sum_{i_1+i_2=i} w_{i_1}(V)\, w_{i_2}(W)$$

Nontriviality:

$$\begin{array}{c} L_{univ} \\ \downarrow \\ \mathbb{R}P^\infty \end{array}$$

Need: $w_1(L_{univ}) \neq 0$

$$L_{univ} = E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathbb{R}^{sign}$$

$$\downarrow$$

$$B\mathbb{Z}/2$$

$$\begin{array}{ccccc} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} S^0 & \longrightarrow & E\mathbb{Z}/2 \times_{\mathbb{Z}/2} D^1 & \longrightarrow & (\mathbb{R}P^\infty)^{L_{univ}} \\ \| & & \simeq & & \\ E\mathbb{Z}/2 & & B\mathbb{Z}/2 & & \end{array}$$

homology iso

$$\tilde{H}^n(B\mathbb{Z}/2) \cong \tilde{H}^n((\mathbb{R}P^\infty)^{L_{univ}})$$
$$\begin{matrix} \Psi & & \Psi \\ x^n & \longleftrightarrow & [L_{univ}]\,x^{n-1} \end{matrix}$$

$$Sq^1(x) = x^2$$

$$Sq^1([L_{univ}]) = x\,[L_{univ}]$$

$$\Rightarrow \quad x = w_1(L_{univ})$$

(Rmk: this also shows $e(L_{univ}) = x$

$\Rightarrow e(L_{univ}) = w_1(L_{univ})$)

Thm

$$H^*(BO(n);\mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \dots, w_n]$$

$$w_i = w_i(V^n_{univ})$$

To prove this, we will need <u>Splitting principle</u>:

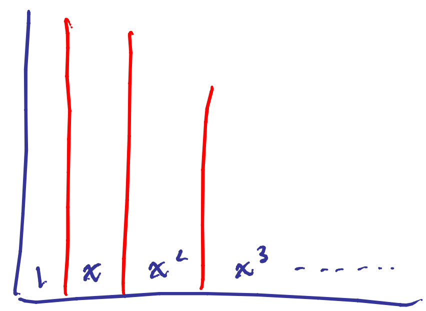
<u>Prop</u>:

$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd} \qquad \begin{tikzcd} V' \oplus L & \\ \pi^* V \arrow[u, phantom, "\cong"] \arrow[r] \arrow[d] & V \arrow[d] \\ P(V) \arrow[r, "\pi"'] & X \end{tikzcd}$$

$\underbrace{\hphantom{P(V)}}_{\text{projective bundle}}$

$$\pi^* : H^*(X) \to H^*(P(V))$$
injective

(pf) <u>SSS</u> $\mathbb{RP}^{n-1} \xrightarrow{\iota} P(V) \xrightarrow{\pi} X$



If x is a P.C., SS collapses, edge homomorphism $\Longrightarrow$ π^* injective

x is a P.C. $\iff$ $x \in H^1(\mathbb{RP}^{n-1})$

$\qquad\qquad\qquad\qquad\qquad\qquad \in$

$\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{Im} \iota^*$

<u>Claim</u>: $\iota^*(w_1(L)) = x$

$$\begin{tikzcd}
\mathbb{RP}^{n-1} \arrow[r, equal] \arrow[d, equal] & P(\mathbb{R}^n) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathbb{RP}^{n-1} \arrow[r] & P(V) \arrow[r] & X
\end{tikzcd}$$

$L_{univ} \to P(\mathbb{R}^n)$, $\quad L \to P(V)$

$\in w_1(L_{univ}) = x$

$\in w_1(L)$, $\quad w_1(L) \mapsto w_1(L_{univ}) = x$

∎

Universal example of splitting principle

$$\begin{tikzcd}
L_{univ}^{\boxtimes n} \arrow[r] \arrow[d] & V^n_{univ} \arrow[d] \\
BO(1)^n \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & BO(n)
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^*BO(n) \arrow[r, hook] & \left(H^*BO(1)^n\right)^{\Sigma_n} \arrow[d, equal] \\
w_i(V^n_{univ}) & & \mathbb{F}_2[x_1,\dots,x_n]^{\Sigma_n} \arrow[d, equal] \\
w_i \arrow[u, mapsto] & & \mathbb{F}_2[e_1,\dots,e_n] \\
 & \mathbb{F}_2[w_1,\dots,w_n] \arrow[uuu] \arrow[ur, "\cong"] &
\end{tikzcd}$$

$$w_i \longmapsto e_i(x_1,\dots,x_n) = w_i(L_{univ}^{\boxtimes n})$$

computed using sum formula

Conclusion: $\mathbb{F}_2[w_1,\dots,w_n] \longrightarrow H^*(BO(n))$

is an iso!

Final rmk:

Prop:

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \quad \dim_{\mathbb{R}} V = n, \qquad w_n(V) = e(V)$$

(pf) case when V = Lsum: done

splitting principle: suffices to prove for

V = sum of lines.

Universal example

$$\begin{array}{c} L_1 \oplus \cdots \oplus L_n \\ \downarrow \\ BO(1)^n \end{array} \qquad e(L_1 \oplus \cdots \oplus L_n) = e(L_1) \cdots e(L_n)$$

$$= x_1 \cdots x_n$$

$$= e_n(x_1, \ldots, x_n)$$

$$= w_n(L_1 \oplus \cdots \oplus L_n)$$

$\square$
