25 - Cohomology of $BO(n)$, Cobordism

Note Title 5/11/2010

Loose end from last lecture:

Nontriviality:

$$\begin{array}{c} L_{univ} \\ \downarrow \\ \mathbb{RP}^\infty \end{array}$$

Need: $w_1(L_{univ}) \neq 0$

$$\begin{array}{c} L_{univ} = E\mathbb{Z}/2 \times_{\mathbb{Z}/2} \mathbb{R}^{sign} \\ \downarrow \\ B\mathbb{Z}/2 \end{array}$$

$$\begin{array}{ccccc} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} S^0 & \longrightarrow & E\mathbb{Z}/2 \times_{\mathbb{Z}/2} D^1 & \xrightarrow{\text{homology iso}} & (\mathbb{RP}^\infty)^{L_{univ}} \\ \| & & \| & & \\ E\mathbb{Z}/2 & & B\mathbb{Z}/2 & & \end{array}$$

$$\tilde{H}^n(B\mathbb{Z}/2) \cong \tilde{H}^n\big((\mathbb{R}P^\infty)^{L_{univ}}\big)$$

$$\begin{array}{ccc} \Psi & & \Psi \\ x^n & \longleftrightarrow & [L_{univ}]\, x^{n-1} \end{array}$$

$$Sq^1(x) = x^2$$

$$Sq^1([L_{univ}]) = x\,[L_{univ}]$$

$$\Rightarrow \quad x = w_1(L_{univ})$$

(Rmk: this also shows $e(L_{univ}) = x$

$\Rightarrow e(L_{univ}) = w_1(L_{univ})$)

Thm

$$H^*(BO(n); \mathbb{F}_2) \cong \mathbb{F}_2[w_1, w_2, \ldots, w_n]$$

$$w_i = w_i(V^n_{univ})$$

To prove this, we will need <u>Splitting principle</u>:

<u>Prop</u>:

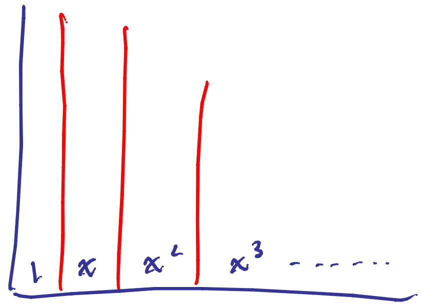
$$\begin{tikzcd} V \arrow[d] \\ X \end{tikzcd} \qquad \begin{tikzcd} V' \oplus L \arrow[d, phantom, "\cong" rotate=90] & \\ \pi^* V \arrow[r] \arrow[d] & V \arrow[d] \\ P(V) \arrow[r, "\pi"'] & X \end{tikzcd}$$

$\underbrace{}_{\text{projective bundle}}$

$$\pi^* : H^*(X) \to H^*(P(V))$$

injective

(pf) <u>SSS</u> $\mathbb{RP}^{n-1} \xrightarrow{\iota} P(V) \xrightarrow{\pi} X$



If x is a P.C., SS collapses, edge homomorphism $\Longrightarrow$ π^* injective

x is a P.C. $\Longleftrightarrow$ $x \in H^1(\mathbb{RP}^{n-1})$
$\quad\in$
$\quad\mathrm{Im}\ \iota^*$

<u>Claim</u>: $\iota^*(w_1(L)) = x$

$$\begin{tikzcd}
\mathbb{RP}^{n-1} \arrow[rr, equal] \arrow[dd, equal] & & P(\mathbb{R}^n) \arrow[r] \arrow[dd] & * \arrow[dd] \\
 & L_{univ} \arrow[ur] \arrow[dd] & & \\
\mathbb{RP}^{n-1} \arrow[rr] & & P(V) \arrow[r] & X \\
 & L \arrow[ur] & &
\end{tikzcd}$$

$\in \ w_1(L_{univ}) = x$

$\in \ w_1(L)$

□

Universal example of splitting principle

$$\begin{tikzcd} L_{univ}^{\boxplus n} \arrow[r] \arrow[d] & V_{univ}^{n} \arrow[d] \\ BO(1)^{n} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & BO(n) \end{tikzcd}$$

$$\begin{tikzcd} & H^*BO(n) \arrow[r, hook] & \left(H^*BO(1)^n\right)^{\Sigma_n} \arrow[d, equal] \\ w_i(V_{univ}^n) & & \mathbb{F}_2[x_1, \dots, x_n]^{\Sigma_n} \arrow[d, equal] \\ w_i \arrow[u, mapsto] & \mathbb{F}_2[w_1, \dots, w_n] \arrow[uu] \arrow[r, "\cong"] & \mathbb{F}_2[e_1, \dots, e_n] \end{tikzcd}$$

$$w_i \longmapsto e_i(x_1, \dots, x_n) = w_i(L_{univ}^{\boxplus n})$$

computed using axioms

Conclusion: $\mathbb{F}_2[w_1, \dots, w_n] \longrightarrow H^*(BO(n))$

is an iso!

Final remk:

Prop: $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ $\dim_{\mathbb{R}} V = n$, $\qquad w_n(V) = e(V)$

(pf) case when V = line : done

Splitting principle: suffices to prove for

V = sum of lines.

universal example

$$\begin{array}{c} L_1 \oplus \cdots \oplus L_n \\ \downarrow \\ BO(1)^n \end{array} \qquad e(L_1 \oplus \cdots \oplus L_n) = e(L_1) \cdots e(L_n)$$

$$= x_1 \cdots\cdots x_n$$

$$= e_n(x_1, \ldots, x_n)$$

$$= w_n(L_1 \oplus \cdots \oplus L_n)$$

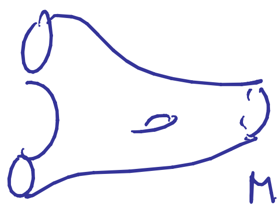
$\square$

Cobordism

M_1, M_2 = closed smooth d-dim'l mflds

Cobordant if $\exists$ mfld w/ ∂ W (dim $d+1$)

$$\partial W = M_1 \amalg M_2$$



M_2

M_1

$$\Omega_d = \left\{ \begin{matrix} \text{diffeo classes of} \\ \text{closed } d\text{-manifolds} \end{matrix} \right\} \Big/ \text{cobordism}$$

Ω_* is a graded ring

$$[M_1] + [M_2] = [M_1 \amalg M_2]$$

$$0 = [\emptyset]$$

$$\partial(M \times I) = M \amalg M \Rightarrow 2[M] = 0$$

$$\Rightarrow [M] = [-M]$$

$$[M_1][M_2] = [M_1 \times M_2]$$

$$* = [1]$$

Problem: compute Ω_*

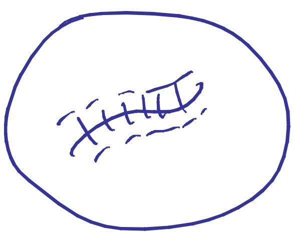
Pontriagin – Thom construction:

M = closed d-mfld

$M \overset{\iota}{\hookrightarrow} \mathbb{R}^{d+N}$ $\quad N \gg 0$, ν_N = normal bundle

$$S^{d+N} \longrightarrow M^{\nu_N} \longrightarrow BO(N)^{\gamma^N_{univ}} = MO(N)$$

collapse everything outside tubular nbhd to pt



get $[M, \iota] \in \pi_{d+N} MO(N)$

Def: $\pi_d MO := \varinjlim_N \pi_{d+N} MO(N)$

$$\Sigma MO(N) = BO(N)^{\gamma^N \oplus \mathbb{R}} \longrightarrow BO(N+1)^{\gamma^{N+1}} = MO(N+1)$$

$$(S^{d+N} \to MO(N)) \mapsto (S^{d+N+1} \to MO(N+1))$$

get $\langle M, \iota \rangle \in \pi_d(MO)$

(1) $\langle M, \iota \rangle$ is independent of embedding ι

$$\begin{tikzcd} M \arrow[r, hook, "\iota_N"'] \arrow[rr, hook, bend right, "\iota_{N+1}"'] & \mathbb{R}^{d+N} \arrow[r, hook] & \mathbb{R}^{d+N+1} \end{tikzcd}$$

$$\begin{tikzcd} \langle M, \iota_N \rangle \in \pi_{d+N} MO(N) \arrow[d, mapsto] \arrow[d] \\ \langle M, \iota_{N+1} \rangle \in \pi_{d+N+1} MO(N+1) \end{tikzcd}$$

$N \gg 0$, any two embeddings are smoothly isotopic

$\Rightarrow \langle M, \iota \rangle \simeq \langle M, \iota' \rangle$

(equal in $\pi_{d+N} MO(N)$)

Consequence $\langle M \rangle \in \pi_d (MO)$

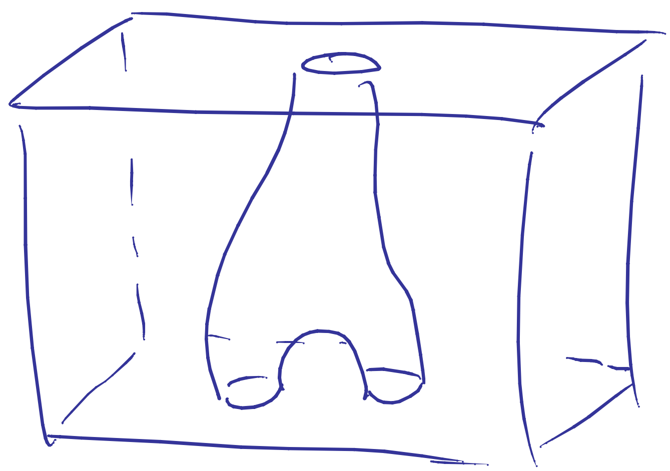
(2) $\partial W = [M_1] \amalg [M_2]$

$\Rightarrow \langle M_1 \rangle = \langle M_2 \rangle$

embed $W \underset{\iota}{\hookrightarrow} \mathbb{R}^{d+N} \times I$

s.t. $W \cap \mathbb{R}^{d+N} \times \{0\} = M_1$

$W \cap \mathbb{R}^{d+N} \times \{1\} = M_2$



$\rightsquigarrow$ $\iota_1 : M_1 \hookrightarrow \mathbb{R}^{d+N}$

$\iota_2 : M_2 \hookrightarrow \mathbb{R}^{d+N}$

$\Longrightarrow$ Pont-Thom collapse

$$S^{d+N} \wedge I_+ \longrightarrow W^\nu \longrightarrow MO(N)$$

homotopy between $\langle M_1, \iota_1 \rangle$ and $\langle M_2, \iota_2 \rangle$

Get: $\Omega_* \longrightarrow \pi_* MO$

(check this is a map of ab gps)

(in fact $\pi_* MO$ is a ring

$$BO(n)^{V_n} \wedge BO(m)^{V_m} \longrightarrow BO(n+m)^{V_{n+m}}$$

$$\shortparallel$$

$$\left(BO(n) \times BO(m)\right)^{V_n \boxplus V_m}$$

$$MO(n) \wedge MO(m) \longrightarrow MO(n+m)$$

$$\Longrightarrow \quad \pi_{d_1} MO \otimes \pi_{d_2} MO \longrightarrow \pi_{d_1+d_2} MO$$

$$\Omega_* \longrightarrow \pi_* MO \quad \text{ring map.})$$