3 – Pointed spaces and htpy gps

Note Title 2/9/2010

Last time: Redefined Top, $X \times Y$, $\underline{Map}(X,Y)$

s.t.

$$Map(X \times Y, Z) \xrightarrow[\cong]{} Map(X, \underline{Map}(Y,Z))$$

Consequence:

Prop:

$$[X,Y] = \pi_0 \underline{Map}(X,Y)$$

pf

$$\begin{array}{ccc} \text{homotopies} & \longleftrightarrow & \text{paths in } \underline{Map}(X,Y) \\ \parallel & & \parallel \\ \{X \times I \to Y\} & \longleftrightarrow & \{I \longrightarrow \underline{Map}(X,Y)\} \end{array}$$

$\square$

$Top_* =$ ctgy objects $= \{(X,x) \mid x \in X \in Top\}$

morphisms = bspt preserving

[abusively: $* \in X$ will denote bspt.]

Defn

$$\underline{Map}_*(X,Y) \subseteq \underline{Map}(X,Y)$$

Top_*

subspace of pointed maps

Def smash product $X, Y \in \mathrm{Top}_*$

$$X \wedge Y := \frac{X \times Y}{X \times * \cup * \times Y} \in \mathrm{Top}_*$$

bspt: collapse point

the following explains why smash product is important

Prop: there is an adjunction

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \underline{\mathrm{Map}}_*(Y, Z))$$

(pf)

$$\begin{tikzcd}
X \times * \cup * \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \times Y \arrow[d] \arrow[ddr, bend left, "f"] & \\
* \arrow[r] \arrow[drr, bend right] & X \wedge Y \arrow[dr, dashed] & \\
& & Z
\end{tikzcd}$$

$$f \longleftrightarrow \tilde{f} : X \longrightarrow \underline{\mathrm{Map}}(Y, Z)$$

(i) $\left[\begin{array}{l} x \longmapsto (* \longmapsto \underset{\substack{\| \\ *}}{f(x, *)}) \\ \iff \tilde{f}(x) \in \underline{\mathrm{Map}}_*(Y, Z) \end{array}\right.$

(ii) $\left[\begin{array}{l} * \longmapsto (y \longmapsto \underset{\substack{\| \\ *}}{f(*, y)}) \\ \iff \tilde{f} \text{ pointed} \end{array}\right.$

$\square$

There is an adjunction:

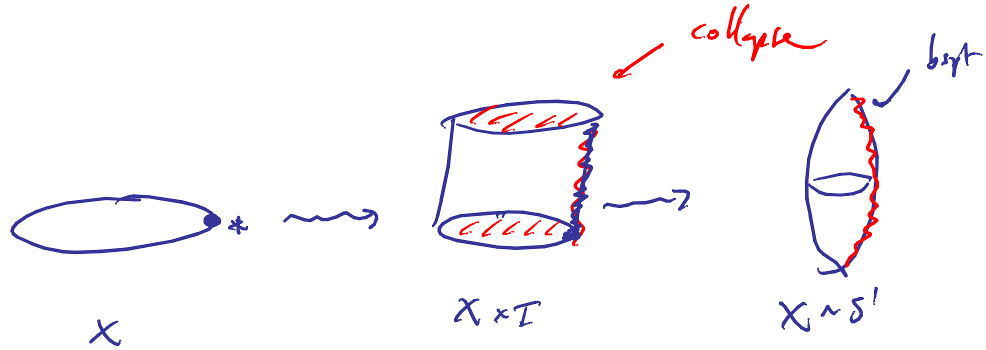
$$(-)_+ : \mathrm{Top} \rightleftarrows \mathrm{Top}_* : \text{forget}$$

X_+ = add disjoint basept

$$\mathrm{Map}_*(X_+, Y) \cong \mathrm{Map}(X, Y)$$

<u>Def</u>: <u>suspension</u>

$$\Sigma : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$$
$$X \longmapsto X \wedge S^1$$



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e.g. $X = S^1$

$\Sigma X \approx S^2$

In general $\Sigma S^n \approx S^{n+1}$

Also

$$X \wedge S^0 \approx S^0$$

$$X_+ \wedge Y_+ \approx (X \times Y)_+$$

$f, g : X \longrightarrow Y$ pointed maps

A pointed htpy $H : X \times I \longrightarrow Y$

satisfies $H_t : X \to Y$ pointed

i.e. $\tilde{H} : I \longrightarrow \underline{\mathrm{Map}}_*(X, Y)$

not a pointed map

$\updownarrow$

$I_+ \longrightarrow \underline{\mathrm{Map}}_*(X, Y)$

pointed

$\updownarrow$

$$\boxed{H : X \wedge I_+ \longrightarrow Y}$$

pointed homotopies are conveniently expressed this way.

$[X, Y]$ = htpy classes of maps $X \to Y$

$[X, Y]_*$ = pointed htpy classes of maps

ΩX = based loop space.

<u>Def</u>: $\Omega : Top_* \longrightarrow Top_*$

$X \longmapsto \underline{Map}_*(S^1, X)$ "space of loops"

<u>Note</u>: $\pi_1 X = \pi_0 \Omega X$

There is an adjunction

$$\Sigma : Top_* \rightleftarrows Top_* : \Omega$$

$$\begin{array}{ccc} \Sigma X \longrightarrow Y & \longleftrightarrow & X \longrightarrow \Omega Y \\ \shortparallel & & \shortparallel \\ X \wedge S^1 & & \underline{Map}_*(S^1, Y) \end{array}$$

<u>Check</u>: $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$

$\underline{Top_*}$ complete, cocomplete

$X \in Top_*^{\,I}$

$$\varprojlim^{Top_*} X = \varprojlim^{Top} X$$

$$\shortparallel$$

$$\left\{ (x_i)_{i \in I} \;\middle|\; \begin{array}{l} x_i \in X(i) \\ \forall\, \alpha : i \to j \\ \alpha_*(x_i) = x_j \end{array} \right\} \longrightarrow (*)_{i \in I} \in \varprojlim X$$

bspt.

$$\varinjlim^{Top_*} X = \coprod_{i \in I} X(i) \Big/ \sim$$

$\forall\ \alpha : i \to j$

$x_i \in X(i) \qquad x_i \sim \alpha_*(x_i)$

and $\forall\ i, j$

$*_{X(i)} \sim *_{X(j)}$

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e.g. coproduct

$$X \amalg_{Top^*} Y = X \vee Y \qquad \text{"wedge"}$$

$$X \vee Y = X \amalg_{Top} Y \Big/ *_X \sim *_Y$$

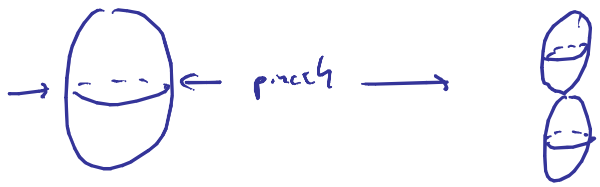
Homotopy grps

$n \geq 1$

$$\pi_n(X) = [S^n, X]_*$$

π_n is a group:

$$S^n \longrightarrow S^n \vee S^n$$



$$\alpha : S^n \longrightarrow X$$

$$\beta : S^n \longrightarrow X$$

$$\alpha+\beta : S^n \xrightarrow[\text{pinch}]{} S^n \vee S^n \xrightarrow[\alpha\vee\beta]{} X$$

$$e : S^n \longrightarrow * \hookrightarrow X$$

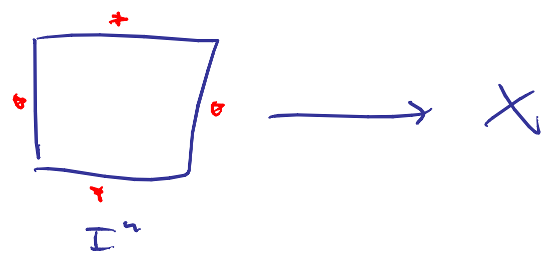
Note: $\pi_0(X) = [S^0, X]_*$

$\pi_0(X)$ is only a pointed set

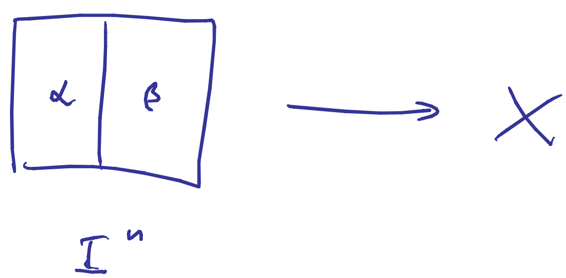
Alternate description

$$\pi_n(X) = [(I^n, \partial I^n), (X, *)]$$

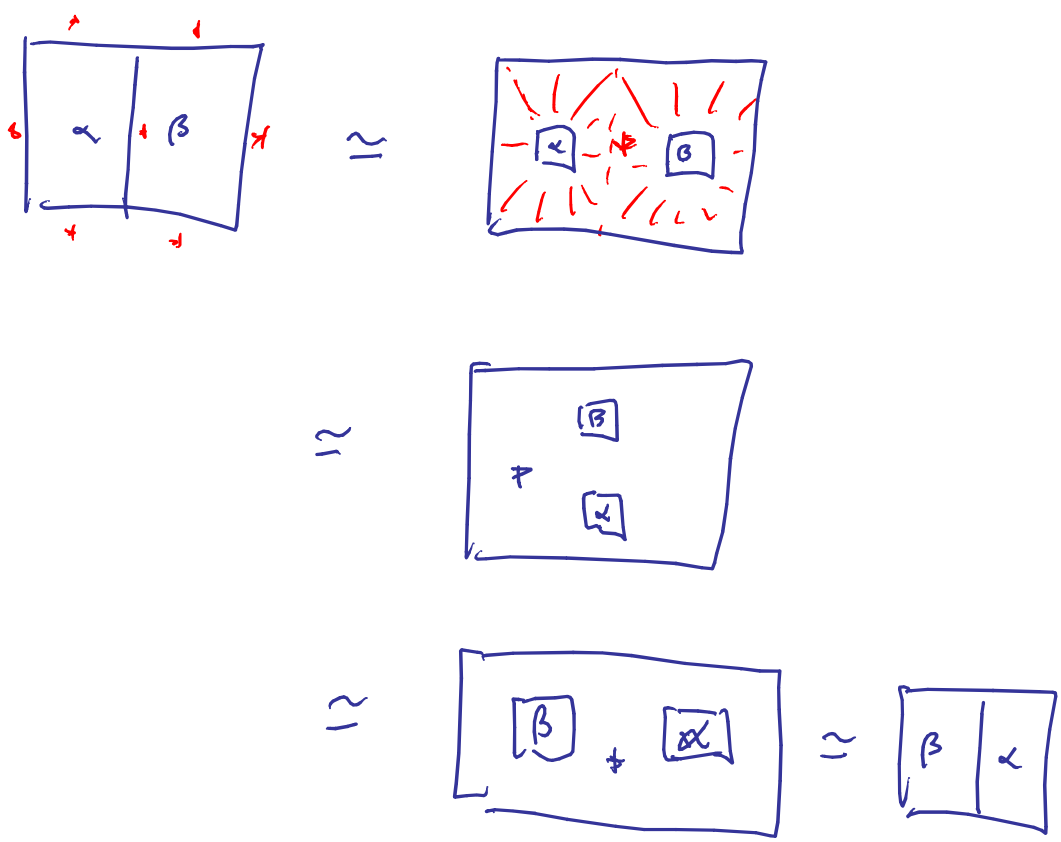
$\alpha \in \pi_n X$



$\alpha + \beta$



prop: $k > 1 \Rightarrow \pi_n X$ abelian



∎

$f \in \mathrm{Map}_*(X, Y)$

$\Longrightarrow \quad f_* : \pi_n X \longrightarrow \pi_n Y$

$\alpha \longmapsto f \circ \alpha$

$\pi_n : \mathrm{Top}_* \longrightarrow \mathrm{Grp}$ is a functor.

<u>Prop:</u> $\pi_{<n}(S^n) = 0$

pf: smooth approximation

<u>Prop:</u> $p : X \longrightarrow Y$ covering space

$\Longrightarrow \quad p_* : \pi_k(X) \longrightarrow \pi_k(Y)$

is
- monomorphism $k = 1$
- isomorphism $k \geq 2$

($k = 1$: covering space theory)

Pf Since S^n is simply connected

$(n \geq 2)$

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ S^n \arrow[r, "\alpha"'] \arrow[ur, dashed, "\tilde{\alpha}", "\exists!"'] & Y \end{tikzcd}$$

$\Rightarrow$ p_* surjective

Furthermore given a homotopy $H : p_*(\beta) \simeq p_*(\beta')$

HLP lifting property:

$$\begin{tikzcd} S^n \wedge \{0\}_+ \arrow[r] \arrow[d] & X \arrow[d] \\ S^n \wedge I_+ \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}", "\exists!"'] & Y \end{tikzcd}$$

simply connected

uniqueness of lifts $\Rightarrow$ $\tilde{H}_1 = \beta'$

$\Rightarrow$ p_* injective

$\square$

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ex: $\mathbb{R} \longrightarrow S^1$ covering

$$\Rightarrow \quad \pi_k(S^1) = \begin{cases} \mathbb{Z}, & k = 1 \\ 0, & k \neq 1 \end{cases}$$