5 - cofibrations

Note Title 2/11/2010

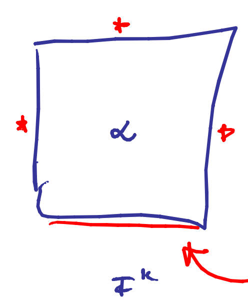
<u>Relative Htpy gps:</u>

$$A \hookrightarrow X \quad \text{in } \mathrm{Top}_*$$

$$\pi_k(X,A) = \left[\left(I^k, \partial I^k, \partial I^k - I^{k-1}\times\{0\}\right), (X, A, *) \right]$$

e.g. $[\alpha] \in \pi_k(X,A)$

$\alpha \sim$



one face only maps into A

rest of faces map to $*$

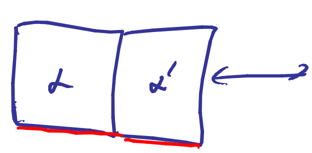
only works for $k > 0$!

$k = 1$: this is a pointed set

$k \geq 2$: this is a gp

$k \geq 3$: abelian

addition



$\alpha' + \alpha$

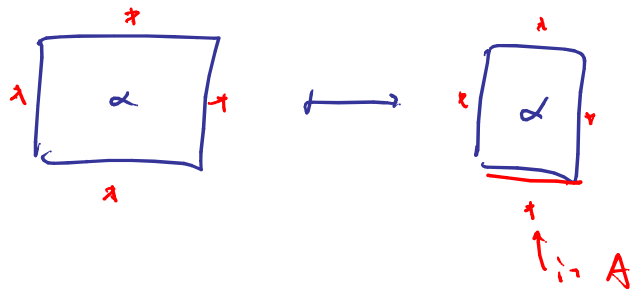
LES of a pair:

$$i : A \hookrightarrow X$$

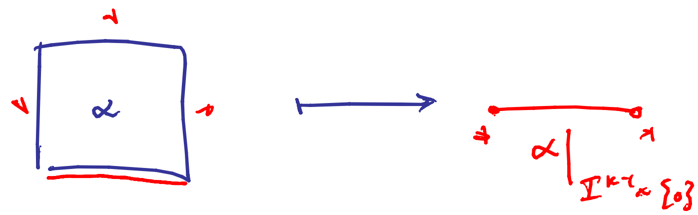
Then $\exists$ a LES:

$$\cdots \xrightarrow{\partial} \pi_k(A) \xrightarrow{i_*} \pi_k(X) \xrightarrow{j} \pi_k(X,A) \xrightarrow{\partial} \pi_{k-1}(A) \xrightarrow{i_*} \cdots$$

j:



∂:



Sequence ends w/

$$\pi_1(A) \xrightarrow{i_*} \pi_1(X) \xrightarrow{\partial} \pi_1(X,A) \xrightarrow{j} \pi_0(A) \xrightarrow{i_*} \pi_0(X)$$

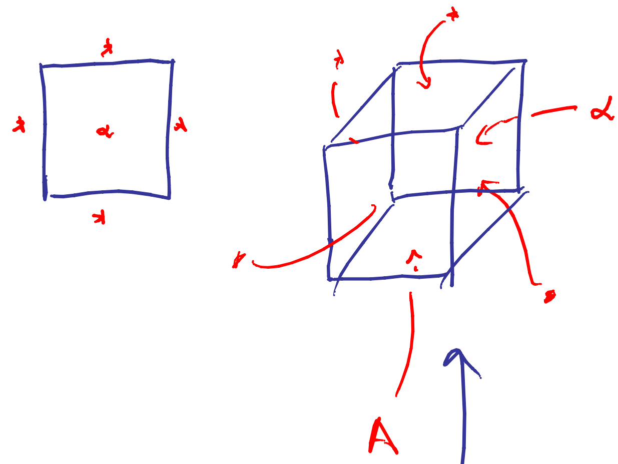
pointed sets

ker = im

w/ "ker" = preimage of bspt

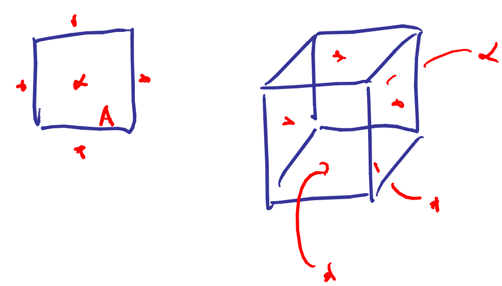
(Sketch)

$$\pi_k(A) \longrightarrow \pi_k(X) \longrightarrow \pi_k(X,A)$$



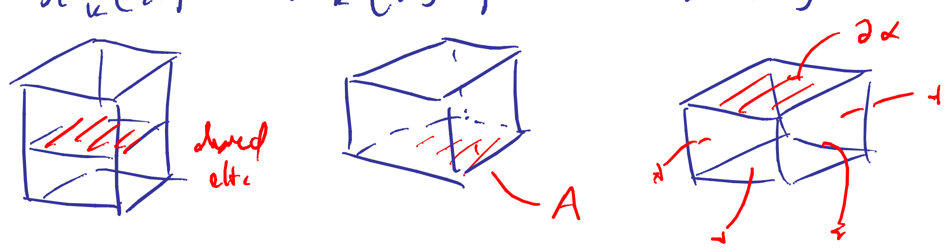
gives htpy between α
and elt of $\pi_k(A)$

$$\pi_k(X,A) \longrightarrow \pi_{k-1}(A) \longrightarrow \pi_{k-1}(X)$$



this is an elt
of $\pi_k(X,A)$

$$\pi_k(X) \longrightarrow \pi_k(X,A) \longrightarrow \pi_{k-1}(A)$$



Whitehead product (won't use these)

$S^k \times S^n$ has CW cx

0-cell

k-cell

n-cell

k+n-cell

$$\begin{tikzcd} S^{k+n-1} \arrow[r, "a_{k,n}"] \arrow[d] & S^k \vee S^n \arrow[d] \\ D^{k+n} \arrow[r] & S^k \times S^n \end{tikzcd}$$

$\alpha \in \pi_k(X)$

$\beta \in \pi_n(X)$, $[\alpha,\beta] \in \pi_{k+n-1}(X)$

$$[\alpha,\beta] : S^{k+n-1} \longrightarrow S^k \vee S^n \xrightarrow[\alpha \vee \beta]{} X$$

"$\pi_*(X)$ is a graded Lie algebra"

Cofibrations and fibrations

"Eckmann-Hilton" Duality

Table

H^*	π_*
LES pair	LES pair
$j: A \to X$ cof $\Rightarrow H^*(X,A) \cong \tilde{H}^*(X/A)$	$f: E \to B$ fibration, $F = f^{-1}(*)$ $\Rightarrow \pi_*(E,F) \cong \pi_{*-1}(B)$
cofiber sequence	fiber sequence
$\tilde{H}^*(\Sigma X) \cong \tilde{H}^{*-1}(X)$	$\pi_* \Omega X \cong \pi_{*+1} X$
cup product (graded commutative)	Whitehead product (graded Lie)

likes good colimits

likes good limits

Cofibrations

$i : A \hookrightarrow X$ inclusion of subcx into a CW-cx

$$\Rightarrow \quad H^*(X,A) \cong \tilde{H}^*(X/A)$$

We want a general class of inclusions that satisfies this.

Def! A map $i: A \longrightarrow X$ is a cofibration if it satisfies the htpy extension property:

$\forall \; f : X \longrightarrow Y$

$\forall \; H : A \longrightarrow \underline{\mathrm{Map}}(I,Y)$ making the square commute

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "i"'] & \underline{\mathrm{Map}}(I,Y) \arrow[d, "ev_0"] \\
X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{H}"] & Y
\end{tikzcd}$$

$\exists \; \tilde{H}$ as above making diagram commute.

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i.e. "for every $f : X \to Y$ and every homotopy $H : f|_A \simeq g$ $\exists \; \tilde{g} : X \to Y$ and $\tilde{H} : f \simeq \tilde{g}$ extending H."

Rmk:

(1) cofibrations are necessarily closed inclusions

(2) $A \hookrightarrow X$ is a cofibration iff (X,A) is an "NDR pair"

Neighborhood
Deformation
Retract.

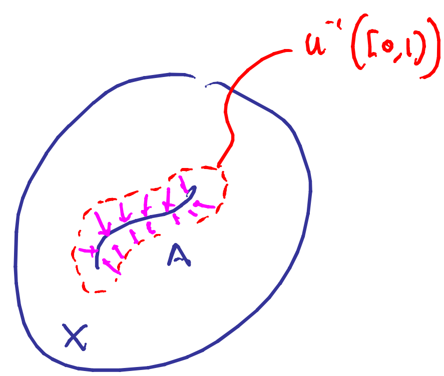
Def (X,A) is an NDR pair $(A \subseteq X)$ if

$$\exists \ X \xrightarrow[u]{} I \qquad u^{-1}(0) = A$$

$$h : X \times I \longrightarrow X$$

$$\begin{tikzcd}
 & X\times\{0\} \arrow[d] \arrow[dr, Rightarrow] & \\
u^{-1}([0,1))\times\{1\} \arrow[r, hook] \arrow[drr, bend right] & X\times I \arrow[r, "h"] & X \\
 & A\times I \arrow[u, hook] \arrow[r, "\pi"] & A \arrow[u, hook]
\end{tikzcd}$$

picture

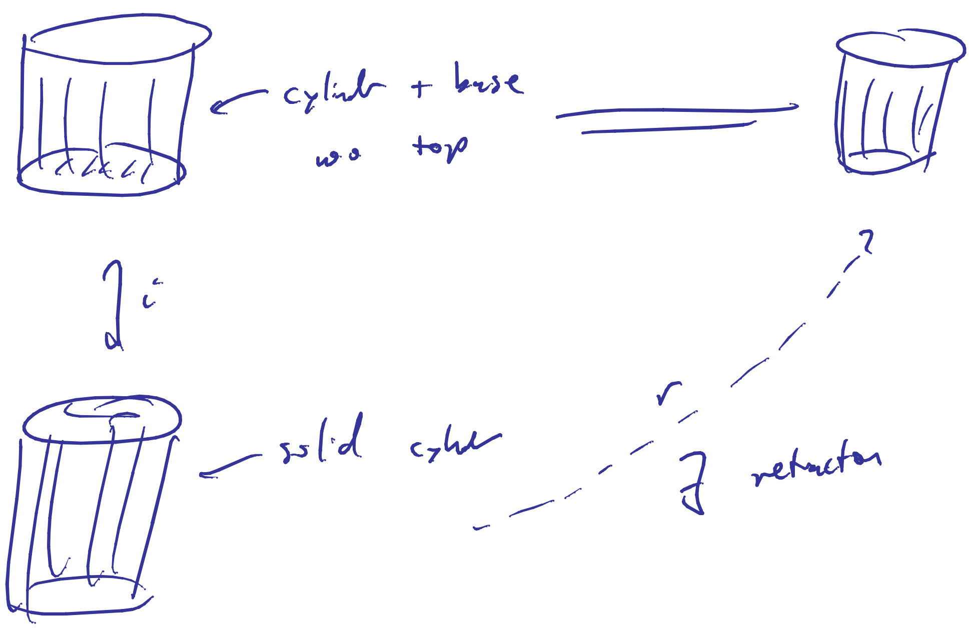


e.g. closed submanifolds, tubular nbhds give cofibrations

e.g. $S^{n-1} \hookrightarrow D^n$ is a cofibration

Different proof

$$
\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "i"] & \underline{\mathrm{Map}}(I, Y) \arrow[d] \\
D^n \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Map}}(\{0\}, Y)
\end{tikzcd}
\quad \longleftrightarrow \quad
\begin{tikzcd}
S^{n-1} \times I \underset{S^{n-1}\times\{0\}}{\cup} D^n \times I \arrow[r, "f"] \arrow[d] & Y \\
D^n \times I \arrow[ur, dashed, "f \circ r"'] &
\end{tikzcd}
$$



composition of cofibs are cofibs, Id is a cofib

lemma pushouts of cofibrations are cofibrations

lemma $\emptyset \longrightarrow X$ cofibration

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{cofib}"'] & A \amalg Y \arrow[d] \\ X \arrow[r] & X \amalg Y \end{tikzcd}$$

$\amalg$ cofib

$\Rightarrow$ $A \amalg B \longrightarrow X \amalg Y$ cofib

lemma $X_0 \xrightarrow[\alpha_0]{} X_1 \xrightarrow[\alpha_1]{} \cdots$ sequence of cofibrations

$\Longrightarrow$ $X_0 \longrightarrow \varinjlim X_i$ is a cofib.

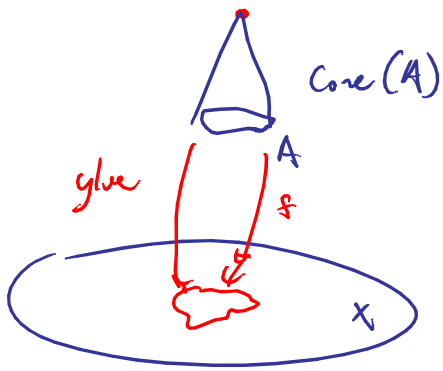
$\Longrightarrow$ inclusions of CW cx's are cofibs.

<u>unreduced Mapping Cone</u>:

$A \xrightarrow[f]{} X$ map

define

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_0"] & X \arrow[d] \\ A \times I \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$



excision

$$\Rightarrow \tilde{H}^*(\mathrm{Cone}(A)) \cong H^*(X, A)$$

In Homework: you show

$A \hookrightarrow X$ cofibration

$$\Rightarrow \quad \mathrm{Cone}(A) \longrightarrow X/A$$

is a htpy equivalence

(deduce that $\tilde{H}^*(X/A) \cong H^*(X, A)$)

Cor: $A \to X$ is a cofibration

$\exists$ L.E.S.

$$\cdots \to \tilde{H}^*(X/A) \longrightarrow H^*(X) \longrightarrow \tilde{H}^*(A) \to \tilde{H}^{*+1}(X/A) \to \cdots$$

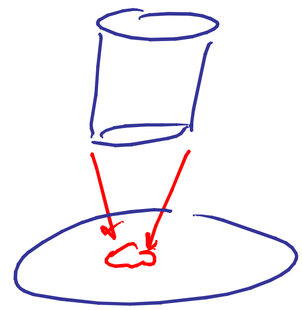
(and reduced version for pointed spaces)

Cofibration = "good quotient"

<u>Mapping cylinders</u>

$f: A \to X$ want to "replace" f w/ a cofibration.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "i_0"] & X \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r, "i_1"] \arrow[rr, bend right, "i"'] & A \times I \arrow[r] \arrow[rr, bend right, "f \circ \pi_A"'] & Cyl(f) \arrow[dr, dashed, "p"] & \\
 & & & X
\end{tikzcd}$$



<u>Lem</u> iy!

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "i"'] & & X \\
 & Cyl(A) \arrow[ur, "p"'] &
\end{tikzcd}$$

i = cofibration

p = htpy equivalence

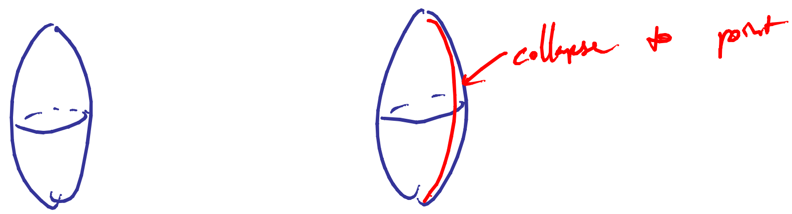
Note $Cyl(A)/A \simeq Cone(f)$

Top_*

pointed versions $\quad X \in \mathrm{Top}_*$

$$\mathrm{Susp}(X) \longrightarrow \Sigma X$$

not nec. a. h.e.



$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ A \wedge I \arrow[r] & C(f) \end{tikzcd}$$

$C(f)$ ← reduced mapping cone or "Cofiber"

$C(A)$ = reduced cone on A

not nec. homotopy equiv.

$$\mathrm{Cone}(f) \longrightarrow C(f)$$

$X \in \mathrm{Top}$

Note $\quad X \xrightarrow{p} *$

$$\mathrm{Cone}(p) = \mathrm{Susp}(X)$$

$$C(p) = \Sigma(X)$$

Def $X \in \mathrm{Top}_*$ is *well pointed* if $* \longrightarrow X$ is a cofibration

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Prop (special case of HW)

If $A \in \mathrm{Top}_*$ is well pointed

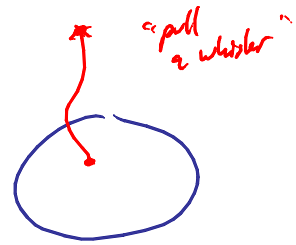
$$f : A \longrightarrow X$$

$$\Longrightarrow \quad \mathrm{Cone}(f) \longrightarrow C(f)$$

is a h.e.

$$\begin{tikzcd}
* \arrow[rr, "i"] \arrow[dr] & & X \\
& \mathrm{Cyl}(X) \arrow[ur, "\text{h.e.}"'] &
\end{tikzcd}$$

$$\mathrm{Cyl}(X) = \tilde{X} \quad \text{well pointed}$$



Consequence for the purposes of htpy theory, can assume every space is well pointed