

8. CW approximation, homology equivalences, lifting excision

Note Title

3/2/2010

Thm (CW approximation)

$$X \in \text{Top}$$

$$\exists \text{ CW cx } \tilde{X}$$

$$\text{v.e. } \tilde{X} \xrightarrow{\cong} X$$

(pf) Suppose X is path connected, pre-LC

\Rightarrow suffices to find based CW cx \tilde{X}

$$\tilde{X} \rightarrow X \quad \text{iso on } \pi_n(-, +) \quad n \geq 1$$

$$\tilde{X}_0 = \bigvee_{\alpha_i} S^{\alpha_i} \quad \alpha_i \text{ real or gens of } \pi_{\geq 1} X$$

$$\tilde{X}_0 \rightarrow X \quad \text{epi on } \pi_0$$

Inductively

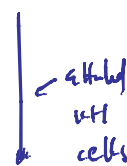
$$\coprod S^k \wedge \partial I_+ \xrightarrow{(\alpha, \beta)} \tilde{X}_{k-1}$$

$$\alpha, \beta \in \pi_k \tilde{X}_{k-1}$$

$$f_{k-1}(\alpha) = f_{k-1}(\beta)$$



$$\coprod S^k \wedge I \xrightarrow{h_{\alpha\beta}} \tilde{X}_k$$



$$\tilde{X}_k$$

$$f_k$$

$$X$$

(homotopy this to factor thru k-skeleton)

*iso on π_n for $n \leq k-1$
epi on π_k*

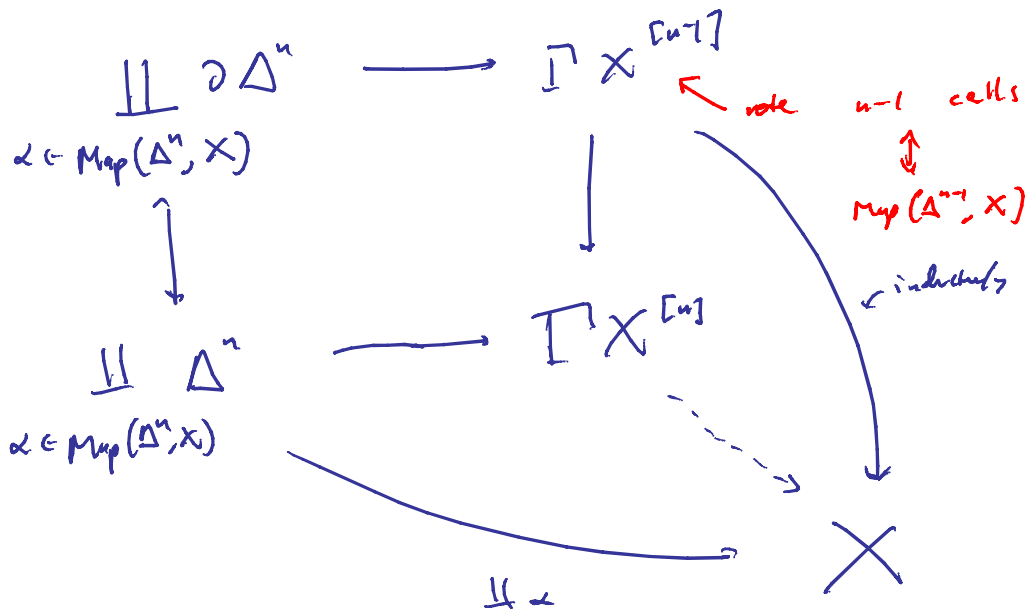
*iso on π_n
epi on π_k*

□

Thm: every w.e. is a homology equiv.

Alternate approach to CW approx.

Define CW via ΓX , $\Gamma X \xrightarrow{f} X$



Note $\Gamma : \text{Top} \rightarrow \text{Top}$ is a functor

Prop: $\Gamma X \rightarrow X$ is a w.e.

(sketch)

Note: n -cells $\Gamma X \leftrightarrow \text{maps } \Delta^n \rightarrow X$

So given $S^n \xrightarrow{\alpha} X$

↑
transfer

$\Rightarrow S^n \xrightarrow{\tilde{\alpha}} \Gamma X$

$\downarrow \quad \downarrow$
 $\downarrow \quad \downarrow$
 $f \quad \cong \quad \pi_0$ -epi

$$S^n \wedge I_+ \xrightarrow{H} X \quad \text{pushed left}$$

↑
+ triangulation

$$\leftarrow \tilde{H} : S^n \wedge I_+ \rightarrow \Gamma X$$

$\Rightarrow f$ is π_1 -mono!

□

lemma $\Gamma X \rightarrow X$ is a homotopy equiv.

(pd) $C_+^{\text{cell}}(\Gamma X) = C_+^{\text{sing}}(X)$

$$C_+^{\text{cell}}(\Gamma X) \xrightarrow{f \text{ iso on } H_0} C_+^{\text{sing}}(\Gamma X)$$

$$\Downarrow f_0 \leftarrow \begin{array}{l} \text{maps} \\ \text{the} \\ \text{rest} \\ \text{to} \\ \text{the} \\ \text{id} \\ \text{on} \\ \text{locally} \end{array}$$

$$C_+^{\text{sing}}(X)$$

pf: w.e. \Rightarrow homby iso.

$$X \rightarrow Y \quad \text{w.e.}$$

$$\begin{array}{ccc} \Gamma X & \xrightarrow[\text{w.e.}]{H_0 \cong} & X \\ \downarrow & & \downarrow \text{w.e.} \\ \Gamma Y & \xrightarrow[\text{w.e.}]{H_0 \cong} & Y \end{array}$$

$$\Rightarrow \Gamma X \rightarrow \Gamma Y \quad \text{v.e.}$$

$$\Rightarrow \Gamma X \rightarrow \Gamma Y \quad \text{h.e.}$$

$$\Rightarrow \Gamma X \rightarrow \Gamma Y \quad H_0 - \text{iso}$$

$$\Rightarrow X \rightarrow Y \quad H_0 - \text{iso} \quad \square$$

Homotopy excision

Problem! π_n does not satisfy excision.

$$\left(\text{if it did, } \pi_n X \xrightarrow[\cong]{} \pi_{n+1}(\Sigma X) \right)$$

$$\text{yet } \pi_2(S^1) = 0$$

$$\pi_3(S^2) \neq 0$$

But π_n does satisfy excision through a range....

Thus (Milnor excision)

$$\left. \begin{array}{l} f: A \rightarrow X \quad m\text{-connected} \\ g: A \rightarrow Y \quad n\text{-connected} \end{array} \right\}$$

one of these
is a relative
CW complex

$$\begin{array}{ccccc} F(f) & \longrightarrow & A & \xrightarrow{f} & X \\ \downarrow & & \downarrow g & \downarrow \bar{f} & \downarrow f' \\ F(f') & \longrightarrow & Y & \xrightarrow{f'} & Z \end{array}$$

($m+n$) connected

(pf) Hatcher & May. (we may restrict using SSS)

Cor:

A is connected CW ca

$f: A \rightarrow X$ is n -connected

$$F(f) \longrightarrow \Omega C(f)$$

↑
isomorphism

(pf)

$$\begin{array}{ccccc}
 F(f) & \longrightarrow & A & \xrightarrow[n]{f} & X \\
 \downarrow \text{isom} & & \downarrow \text{in} & & \downarrow \\
 F(i) & \longrightarrow & C(A) & \xrightarrow{i} & C(f) \\
 \downarrow \text{w.e.} & & \downarrow \text{w.e.} & \cong & \downarrow \text{w.e.} \\
 \Omega C(f) & \longrightarrow & * & \longrightarrow & C(f)
 \end{array}$$

e.g.

$$\begin{array}{ccc}
 A & \xleftarrow{\text{cof}} & X \\
 \uparrow i & & \\
 & & \text{is } n\text{-connected}
 \end{array}$$

↑
is n -connected

$$\begin{array}{ccc}
 \pi_k(F(i)) & \longrightarrow & \pi_k(\Omega C(i)) & \text{iso} & k \leq n-1 \\
 \cong & & \cong & & \text{epi } k=n \\
 \pi_{k+1}(X, A) & \longrightarrow & \pi_{k+1}(C(i)) & \cong & \pi_{k+1}(X/A)
 \end{array}$$

So $\pi_0^k(X, A) \rightarrow \pi_0^k(X/A)$
 is iso for $k \leq n+m+1$
 epi for $k = n+m+2$

Cor: (Freudenthal susp. Th.)
 $X = *$

