8. CW approximation, homology equivalences, htpy excision

Note Title 3/2/2010

Thm (CW approximation)

$X \in \mathrm{Top}$

$\exists$ CW cx $\check{X}$

w.e. $\check{X} \xrightarrow{\simeq} X$

(pf) Suppose X is path connected, pick bspt

$\Rightarrow$ suffices to find based CW cx $\check{X}$

$\check{X} \to X$ iso on $\pi_*(-,*)$ $* \geq 1$

$\tilde{X}_0 = \bigvee_{\alpha_i} S^{\alpha_i}$ α_i runs over generators of $\pi_{\geq 1} X$

$\tilde{X}_0 \to X$ epi on π_*

Inductively

(homotop this to factor thru k-skeleton)

$$\begin{tikzcd}
\coprod S^k \wedge \partial I_+ \arrow[r, "(\alpha,\beta)"] \arrow[d] & \tilde{X}_{k-1} \arrow[d, "\text{attached } k+1 \text{ cells}"] \arrow[dr, bend left, "f_{k-1}"] & \\
\coprod S^k \wedge I \arrow[r, "h_{\alpha\beta}"] \arrow[rr, bend right] & \tilde{X}_k \arrow[r, dashed, "f_k"] & X
\end{tikzcd}$$

$\alpha, \beta \in \pi_k \tilde{X}_{k-1}$

$f_k(\alpha) = f_k(\beta)$

f_{k-1}: iso on $\pi_{\leq k-1}$, epi on π_k

f_k: iso on π_k, epi on π_k

$\square$

Thm: every w.e. is a homology equiv.

Alternate approach to CW approx.

Define CW cx ΓX, $\Gamma X \xrightarrow{f} X$

$$
\begin{tikzcd}
\coprod_{\alpha \in Map(\Delta^n, X)} \partial \Delta^n \arrow[r] \arrow[d] & \Gamma X^{[n-1]} \arrow[d] \arrow[ddr, bend left, "\text{inductively}"] & \\
\coprod_{\alpha \in Map(\Delta^n, X)} \Delta^n \arrow[r] \arrow[drr, bend right, "\coprod \alpha"'] & \Gamma X^{[n]} \arrow[dr, dashed] & \\
& & X
\end{tikzcd}
$$

note $n-1$ cells $\updownarrow$ $Map(\Delta^{n-1}, X)$

Note $\Gamma : Top \to Top$ is a functor

Prop: $\Gamma X \to X$ is a w.e.

(pf sketch)
Note: n-cells $\Gamma X \longleftrightarrow$ maps $\Delta^n \to X$

So given $S^n \xrightarrow{\alpha} X$

triangulate

$\Longrightarrow$ $S^n \xrightarrow{\tilde{\alpha}} \Gamma X$

$f_*(\tilde{\alpha}) = \alpha$ $\Rightarrow$ f is π_*-epi

$S^n \wedge I_+ \xrightarrow[H]{} X$ pointed htpy

triangulate

$\longleftrightarrow \quad \tilde{H} : S^n \wedge I_+ \longrightarrow \Gamma X$

$\Longrightarrow \quad f$ is π_*-mono!

$\square$

<u>Lem</u> $\Gamma X \to X$ is a homology equiv.

(pf) $C^{cell}_*(\Gamma X) = C^{sing}_*(X)$

$$\begin{tikzcd}
C^{cell}_*(\Gamma X) \arrow[r, "\text{iso on } H_*"] \arrow[dr, equal] & C^{sing}_*(\Gamma X) \arrow[d, "f_*"] \\
& C^{sing}_*(X)
\end{tikzcd}$$

$\leftarrow$ implies the next be iso on homology pf: w.e. $\Rightarrow$ homology iso.

$X \to Y$ w.e.

$$\begin{tikzcd} \Gamma X \arrow[r, "H_* \cong", "\text{w.e.}"'] \arrow[d] & X \arrow[d, "\text{w.e.}"] \\ \Gamma Y \arrow[r, "H_* \cong", "\text{w.e.}"'] & Y \end{tikzcd}$$

$\Rightarrow$ $\Gamma X \longrightarrow \Gamma Y$ w.e.

$\Rightarrow$ $\Gamma X \longrightarrow \Gamma Y$ h.e.

$\Rightarrow$ $\Gamma X \longrightarrow \Gamma Y$ H_* - iso

$\Rightarrow$ $X \to Y$ H_*-iso $\square$

Homotopy excision

Problem: π_* does not satisfy excision.

(if it did, $\pi_* X \xrightarrow{\cong} \pi_{*+1}(\Sigma X)$)

yet $\pi_2(S^1) = 0$

$\pi_3(S^2) \neq 0$

But π_* does satisfy excision through a range....

Thm (Htpy excision)

$f : A \longrightarrow X$ m-connected

$g : A \longrightarrow Y$ n-connected

one of these is a relative CW cx

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "g'"] \\ F(f') \arrow[r] & Y \arrow[r, "f'"'] & Z \end{tikzcd}$$

$(n+m)$ connected

(pf) Hatcher & May. (we may revisit using SSS)

Cor:

A m connected CW cx

$f : A \to X$ n-connected

$$F(f) \xrightarrow[\uparrow \; n+m \text{ connected}]{} \Omega C(f)$$

(pf)

$$\begin{tikzcd}
F(f) \arrow[r] \arrow[d, "n+m"] & A \arrow[r, "f", "n"'] \arrow[d, "m"] \arrow[dr, phantom, "\ulcorner" very near end] & X \arrow[d] \\
F(i) \arrow[r] \arrow[d, dashed, "w.e."] & C(A) \arrow[r, "i"'] \arrow[d, "w.e."'] \arrow[dr, phantom, "\simeq"] & C(f) \arrow[d, "w.e."] \\
\Omega C(f) \arrow[r] & * \arrow[r] & C(f)
\end{tikzcd}$$

---

e.g.

$$\begin{tikzcd}
A \arrow[r, hook, "cof", "i"'] & X
\end{tikzcd}$$

A m connected, i n-connected

$$\begin{array}{ccc}
\pi_k(F(i)) & \longrightarrow & \pi_k(\Omega C(i)) \\
\wr\wr & & \wr\wr \\
\pi_{k+1}(X, A) & \longrightarrow & \pi_{k+1}(C(i)) \cong \pi_{k+1}(X/A)
\end{array}$$

iso $k \leq n+m$

epi $k = n+m+1$

So $\pi_k(X, A) \longrightarrow \pi_k(X/A)$

is iso for $k \leq n+m+1$

epi for $k = n+m+2$

Cor: (Freudenthal susp. Thm)

$X = *$

$$\begin{tikzcd} A \arrow[r, "f"] & * \arrow[r] & \Sigma A \end{tikzcd}$$

A: m-connected; f: m-connected; $\Sigma A = C(f)$

$F(f) \longrightarrow \Omega\Sigma A$ is $2m$ connected,

$F(f) = A$

adjoint to $\Sigma A \xrightarrow{\mathrm{Id}} \Sigma A$
