9 - $\pi_n S^n$, hurewicz thm

Note Title 3/2/2010

$\pi_k(S^m) \longrightarrow \pi_k(\Omega\Sigma S^{m+1})$ iso $k \le 2m-2$

epi $k = 2m-1$

$m-1$ connected

$\pi_2(S^2) = \mathbb{Z}$ (Hopf fib)

$$\pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \pi_5(S^5)$$

2-connected 4 connected 6 connected

Thm $\pi_n(S^n) \cong \mathbb{Z}$

$$\pi_3(S^2) \xrightarrow{f} \pi_4(S^3) \xrightarrow[\cong]{f} \pi_5(S^4) \xrightarrow[\cong]{f} \cdots$$

f 2-connected, f 4-connected, f 6 connected

$\pi_3(S^2) = \mathbb{Z}$

turns out this is $\mathbb{Z}/2$

$\mathbb{Z}/2$ ...

More generally: fix k

$$\pi_k(S^0) \longrightarrow \pi_{k+1}(S^1) \longrightarrow \pi_{k+2}(S^2) \longrightarrow \pi_{k+3}(S^3) \longrightarrow$$

0-connected 2-connected 4-connected

eventually stabilizes. $\pi_k^s = \varinjlim \pi_{k+n}(S^n)$

$k+i \le 2i-1 \iff i \ge k+2$

get iso

$\pi_0^s = \mathbb{Z}$

e.g. $\pi_1^s = \pi_4(S^3) = \mathbb{Z}/2$

Hurewicz Homo:

$$h : \pi_1(X) \longrightarrow \tilde{H}_1(X)$$

abelianization if X is path connected.

More generally:

get: $h: \pi_k(X) \longrightarrow \tilde{H}_k(X)$

$$f: S^k \longrightarrow X \qquad \tilde{H}_k(S^k) \xrightarrow[f_*]{} \tilde{H}_k(X)$$

$$\iota_k \longmapsto h([f])$$

$f, g \in \pi_k X$

use $f+g : S^k \xrightarrow[\text{pinch}]{} S^k \vee S^k \xrightarrow[f \vee g]{} X$

to get h is a homo.

Relative Hurewicz: $i : A \hookrightarrow X$

$$\begin{array}{ccccc} \pi_k(X,A) & = & \pi_{k-1}(F(i)) & \longrightarrow & \pi_{k-1}(\Omega C(i)) \\ \downarrow & & & & \wr\Vert \\ H_k(X,A) & \cong & \tilde{H}_k(C(f)) & \xleftarrow[h]{} & \pi_k(C(i)) \end{array}$$

Thm: (Hurewicz)

X $n-1$ connected, $n \geq 2$

$\Rightarrow$ $\pi_k X \longrightarrow \tilde{H}_k(X)$ is iso $k = n$

epi $k = n+1$

(pf deferred)

Thm: $f : X \to Y$ $H_\bullet$ - iso

X, Y simply connected

$\Rightarrow$ f is a w.e.

(pf)

$$F(f) \longrightarrow X \longrightarrow Y$$

X, Y simply connected

$$\begin{tikzcd} \pi_2 X \arrow[r] \arrow[d, "\cong"] & \pi_2 Y \arrow[d, "\cong"] \\ \tilde{H}_2 X \arrow[r, "\cong"'] & \tilde{H}_2 Y \end{tikzcd} \Rightarrow \begin{array}{l} F(f) \text{ 1-connected} \\ f \text{ 1-conn} \end{array}$$

$$F(f) \xrightarrow[\;1+1=2 \text{ connected}\;]{} \Omega C(f)$$

$\Rightarrow$ $C(f)$ 2-connected

$\Rightarrow$ $\pi_3 C(f) \xrightarrow[\cong]{} \tilde{H}_3 C(f) = 0$

$\Rightarrow$ $C(f)$ is 3 connected

$\vdots$

$\Rightarrow$ $C(f)$ is π_*-acyclic

So $\pi_2 F(f) \xrightarrow[\cong]{} \pi_2 \Omega C(f)$

$\Rightarrow$ $F(f)$ 2-connected

$\Rightarrow$ $F(f) \longrightarrow \Omega C(f)$

$(2+1) = 3$ connected

$\Rightarrow$ $F(f)$ 3 connected

$\vdots$

$\Rightarrow$ $F(f)$ π_* acyclic

LES $\Rightarrow$ f is π_* iso

(w.e.)

Cor: $f: X \to Y$ simply connected CW cx's

H_* iso $\Longrightarrow$ h.e.

(pf of Hurewicz)

Check: get a map of LES:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k(A) \arrow[r] \arrow[d] & \pi_k(X) \arrow[r] \arrow[d] & \pi_k(X,A) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \tilde{H}_k(A) \arrow[r] & \tilde{H}_k(X) \arrow[r] & \tilde{H}_k(X,A) \arrow[r] & \cdots
\end{tikzcd}$$

X $(n-1)$ connected

Any for cellular approx

CW cx w/ no cells below dim n

$\Longrightarrow$ $\tilde{X} \xrightarrow{\text{w.e.}} X$

Since v.e. $\Rightarrow$ H_*-iso

WLOG X is a CW cx with no cells below dim m.

<u>Case 1</u> $X = \bigvee S^m$

$$\overset{HW}{\Longrightarrow} \pi_m(S^m \vee S^m) \cong \pi_m(S^m) \oplus \pi_m(S^m) \oplus \underbrace{\pi_{m+1}(S^m \times S^m, S^m \vee S^m)}$$

$$\downarrow \; 2m-1 \text{ connected}$$

$$\pi_{m+1}(S^m \wedge S^m)$$

$m+1 \leq 2m-1 \iff 2 \leq m$ ✓

$$\Rightarrow \pi_{m+1}(S^m \times S^m, S^m \vee S^m) = 0$$

this takes care of finite wedges.

use $\varinjlim$ to argue infinite wedges.

$X^{[n]}$ = wedge of spheres .... done. $n+1 > m$

$$\begin{tikzcd} \pi_k X^{[n]} \arrow[r] & \pi_k X^{[n+1]} \arrow[r] & \pi_k(X^{[n+1]}, X^{[n]}) \arrow[d] \\ & & \tilde{H}_k(X^{[n+1]}, X^{[n]}) \end{tikzcd}$$

Hur. exism: $X^{[n]}$ $(m-1)$-connected
$X^{[n+1]}$ $(m-1)$-connected

$$\begin{tikzcd} \pi_k(X^{[n]}, X^{[n+1]}) \arrow[r] \arrow[d, bend right] & \pi_k(X^{[n+1]}/X^{[n]}) \arrow[d, bend left] \\ H_k(X^{[n+1]}, X^{[n]}) \arrow[r, "\cong"'] & \tilde{H}_k(X^{[n+1]}/X^{[n]}) \end{tikzcd}$$

iso for $k \leq 2m-1$

epi for $k = 2m$

$X^{[n+1]}/X^{[n]} = \bigvee S^{n+1}$

iso for $k \leq m+1$
(epi for $k = m+1$) $\Longleftarrow$

iso for

$k \leq n+1$

$\Rightarrow$ iso for $k \leq m+1$

$$
\begin{tikzcd}
 & & k=m \arrow[dll, bend right, "iso"'] \arrow[dl, bend right, "iso"'] \arrow[d, dashed, "iso"] \arrow[dr, bend left, "iso"] \arrow[drr, bend left, "iso"] & & \\
\pi_{k+1}(X^{n+1}, X^{n}) \arrow[r] \arrow[d] & \pi_k(X^{[n]}) \arrow[r] \arrow[d] & \pi_k X^{[n+1]} \arrow[r] \arrow[d] & \pi_k(X^{[n+1]}, X^{[n]}) \arrow[r] \arrow[d] & \pi_{k-1} X^{[n]} \arrow[d] \\
\tilde{H}_{k+1}(X^{n+1}, X^{n}) \arrow[r] & \tilde{H}_k(X^{[n]}) \arrow[r] & \tilde{H}_k(X^{[n+1]}) \arrow[r] & H_k(X^{[n+1]}, X^{[n]}) \arrow[r] & \tilde{H}_{k-1} X^{[n]} \\
 & & k=m+1 \arrow[ul, bend left, "epi"] \arrow[u, dashed, "epi"'] \arrow[ur, bend right, "iso"'] \arrow[urr, bend right, "iso"'] & &
\end{tikzcd}
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