

9 - $\pi_n S^n$, hurewicz thm

$$\pi_k(S^m) \longrightarrow \pi_k(\Omega S^{m+1}) \quad \text{iso } k \leq 2m-2$$

cp: $k = 2m-1$

m-1 *connected*

$$\pi_2(S^2) = \mathbb{Z} \quad (\text{loop fib})$$

$$\pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \pi_5(S^5)$$

\uparrow 2-connected \uparrow 4-connected \uparrow 6-connected

Thm $\pi_n(S^n) \cong \mathbb{Z}$

$$\pi_3(S^2) \xrightarrow{\cong} \pi_4(S^3) \xrightarrow{\cong} \pi_5(S^4) \xrightarrow{\cong} \dots$$

\uparrow 2-connected \uparrow 4-connected \uparrow 6-connected

\mathbb{Z} $\mathbb{Z}/2$ $\mathbb{Z}/2$

was out
was is

More generally: fix k

$$\pi_k(S^0) \longrightarrow \pi_{k+1}(S^1) \xrightarrow{\cong} \pi_{k+2}(S^2) \xrightarrow{\cong} \pi_{k+3}(S^3) \xrightarrow{\cong} \dots$$

\uparrow 0-connected \uparrow 2-connected \uparrow 4-connected

eventually stabilizes.

$$\pi_k^S = \lim_{\longleftarrow} \pi_{k+i}(S^i)$$

$$k+i \leq 2i-1 \iff i \geq k+2$$

set $i=0$

$$\pi_0^S = \mathbb{Z}$$

e.g. $\pi_1^S = \pi_4(S^3) = \mathbb{Z}/2$

Hurewicz Homo:

$$h: \pi_1(X) \longrightarrow \tilde{H}_1(X)$$

abelianization if X is path connected.

More generally:

get: $h: \pi_k(X) \longrightarrow \tilde{H}_k(X)$

$$\begin{array}{ccc} f: S^k \longrightarrow X & & \tilde{H}_k(S^k) \xrightarrow{\delta_*} \tilde{H}_k(X) \\ & & \downarrow \quad \downarrow \\ & & L_k \xrightarrow{\quad} h(f!) \end{array}$$

$$f, g \in \pi_k X$$

use $f+g: S^k \xrightarrow{\text{push}} S^k \vee S^k \xrightarrow{f \vee g} X$

to get h is a homo.

Relative Hurewicz! $c: A \hookrightarrow X$

$$\begin{array}{ccc} \pi_k(X, A) = \pi_{k+1}(F(c)) & \longrightarrow & \pi_{k+1}(\Omega C(c)) \\ \downarrow & & \downarrow \\ H_k(X, A) \cong \tilde{H}_k(c(f)) & \xleftarrow{h} & \pi_k(C(c)) \end{array}$$

Thm! (Morewicz)

X $m-1$ connected, $m \geq 2$

$\Rightarrow \pi_k X \rightarrow \tilde{H}_k(X)$ is iso $k=m$
 epi $k=m+1$

(pf deferred)

Thm! $f: X \rightarrow Y$ H_0 -iso

X, Y simply connected

$\Rightarrow f$ is a w.e.

(pf)

$F(f) \rightarrow X \rightarrow Y$

X, Y simply connected \Rightarrow

$$\begin{array}{ccc} \pi_2 X & \longrightarrow & \pi_2 Y \\ \downarrow \cong & & \downarrow \cong \\ \tilde{H}_2 X & \xrightarrow{\cong} & \tilde{H}_2 Y \end{array} \Rightarrow \begin{array}{l} F(f) \text{ is connected} \\ f \text{ is 1-1} \end{array}$$

$F(f) \xrightarrow{\quad} \Omega C(f)$
 \uparrow
 $H_1 = 2$ connected

$$\Rightarrow C(f) \text{ 2-connected}$$

$$\Rightarrow \pi_3 C(f) \xrightarrow{\cong} \tilde{H}_3 C(f) = 0$$

$$\Rightarrow C(f) \text{ is 3 connected}$$

⋮

$$\Rightarrow C(f) \text{ is } \pi_n\text{-acyclic}$$

so $\pi_2 F(f) \xrightarrow{\cong} \pi_2 \Omega C(f)$

$$\Rightarrow F(f) \text{ 2-connected}$$

$$\Rightarrow F(f) \rightarrow \Omega C(f)$$

(2H) = 3 connected

$$\Rightarrow F(f) \text{ 3 connected}$$

⋮

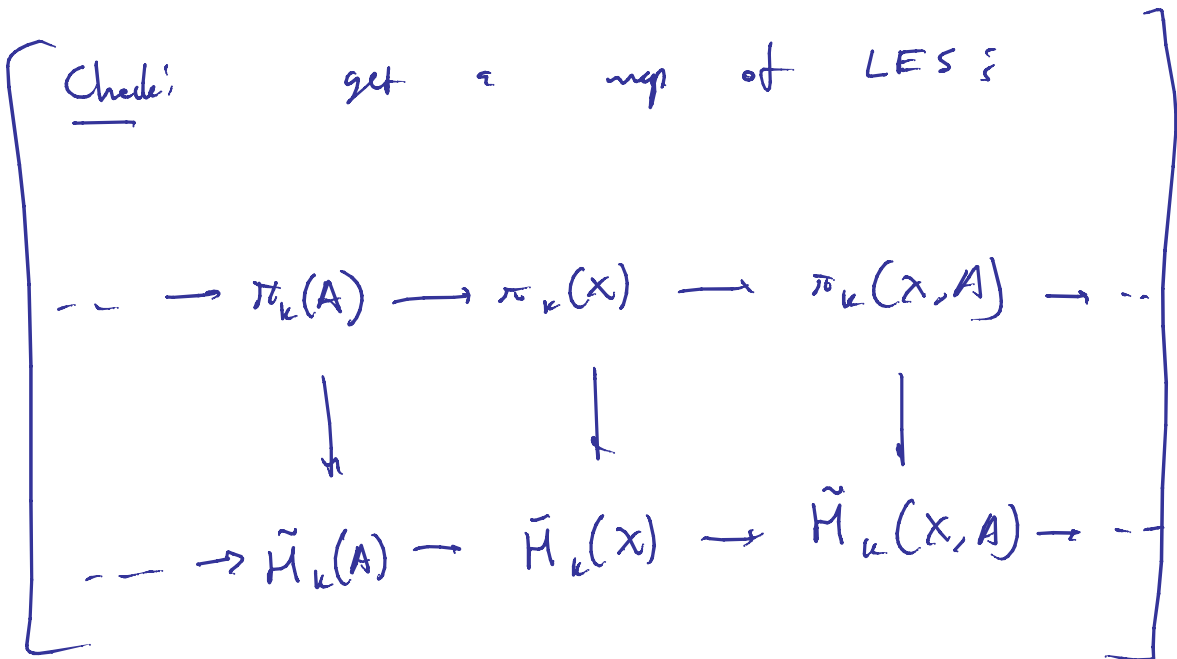
$$\Rightarrow F(f) \text{ } \pi_n\text{-acyclic}$$

LES $\Rightarrow f$ is π_n iso
(w.e.)

Cor: $f: X \rightarrow Y$ simply connected
CW cs

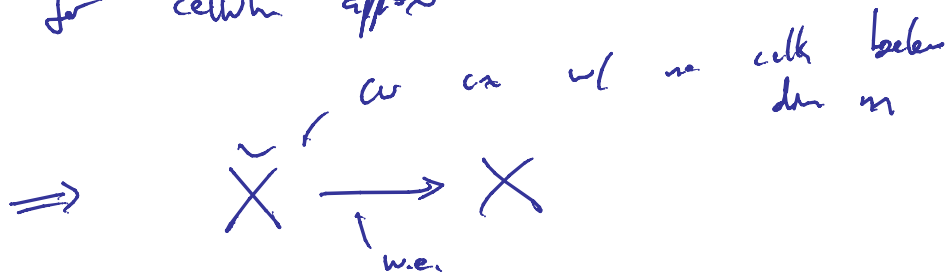
$H_n \cong \mathbb{Z} \Rightarrow$ h.e.

(Pf of Hurewicz)



X $(n-1)$ connected

Arg for cellular approx



Since v.e.c. $\Rightarrow H_0 = 0$

WLOG X is a CW complex
with no cells below dim m .

Case 1 $X = \vee S^m$

HW $\Rightarrow \pi_m(S^m \vee S^m) \cong \pi_m(S^m) \oplus \pi_m(S^m)$

$$\oplus \pi_{m+1}(S^m \times S^m, S^m \vee S^m)$$

\downarrow $2m-1$ connect

$$\pi_{m+1}(S^m \wedge S^m)$$

$$m+1 \leq 2m-1 \Leftrightarrow 2 \leq m \quad \checkmark$$

$$\Rightarrow \pi_{m+1}(S^m \times S^m, S^m \vee S^m) = 0$$

this takes care of finite wedges.

use lim to argue Infinite wedges.

$X^{[m]}$ = wedge of spheres ... dim. $m+1 > m$

$$\pi_k X^{[n]} \longrightarrow \pi_k X^{[n+1]} \longrightarrow \pi_k (X^{[n+1]}, X^{[n]})$$



$$\tilde{H}_k(X^{[n+1]}, X^{[n]})$$

Hzz equivs: $X^{[n]}$ $(m-1)$ -conn'd
 $X^{[n+1]}$ $(m-1)$ -conn'd

$$\pi_k (X^{[n]}, X^{[n+1]}) \longrightarrow \pi_k (X^{[n+1]} / X^{[n]})$$

↑
 iso for $k \leq 2m-1$
 ept for $k = 2m$

$$X^{[n+1]} / X^{[n]} = \bigvee S^{n+1}$$

$$H_k(X^{[n+1]}, X^{[n]}) \xrightarrow{\cong} \tilde{H}_k(X^{[n+1]} / X^{[n]})$$

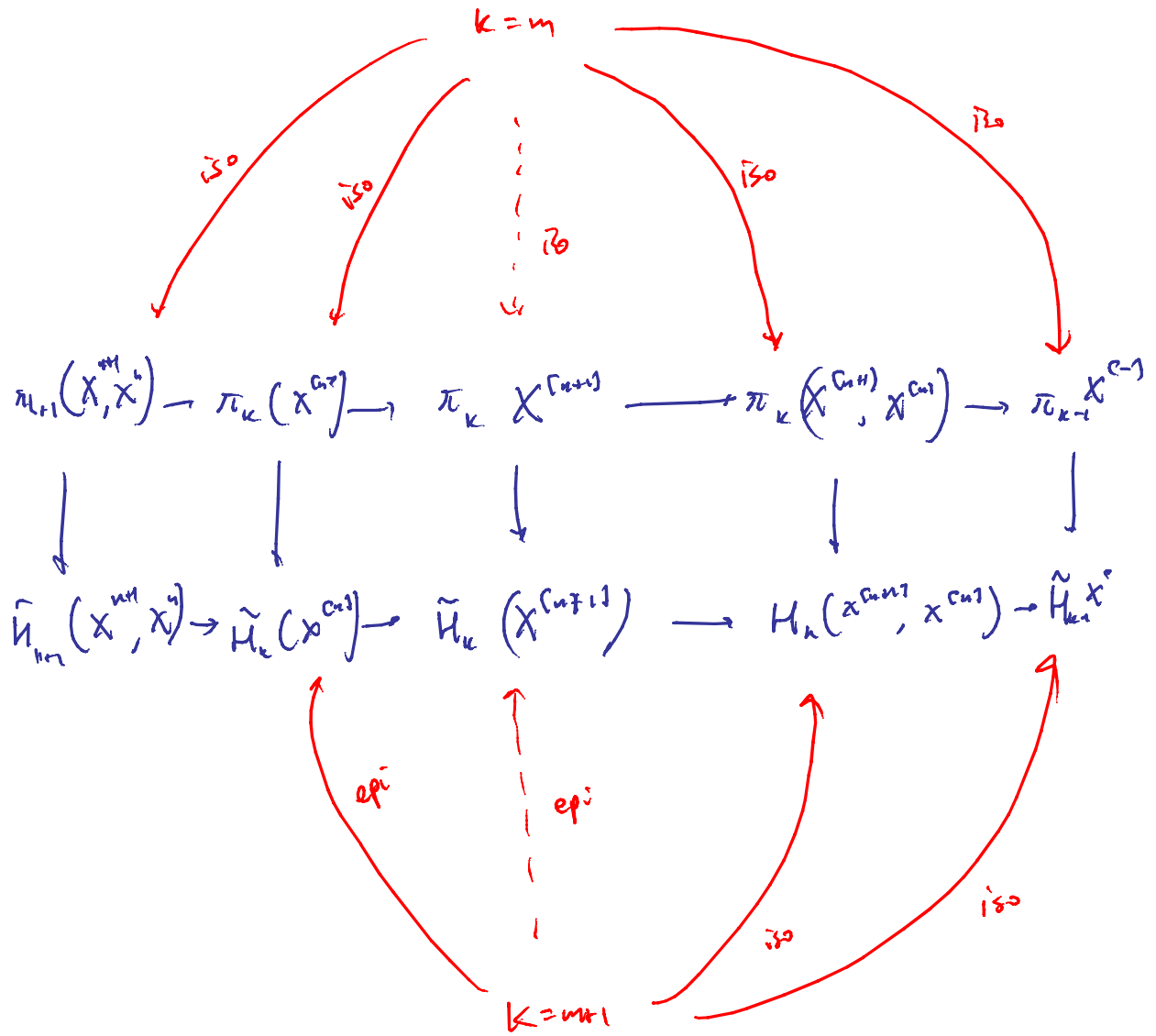
iso for $k \leq m+1$

(e.g. for $k = m+1$)

iso for

$k \leq n+1$

\Rightarrow iso for $k \leq m+1$



□