LECTURE 2: COMPACTLY GENERATED SPACES

References:

- Hatcher, Appendix A.
- May, Chapter 5.
- N. Steenrod, "A convenient category of topological spaces."
- R. Brown, "Function spaces and product topologies."

[We started this lecture by proving that the categories Set and Top are complete and cocomplete by explicitly constructing "formulas" for all limits and colimits]

1. Definitions

Let Top be the category of topological spaces with continuous maps as morphisms. Let $\underline{\mathrm{Map}}(X, Y)$ denote the mapping *space*, with the compact open topology.

The category Top suffers from the fact that the natural map

$$\underline{\mathrm{Map}}(X \times Y, Z) \to \underline{\mathrm{Map}}(X, \underline{\mathrm{Map}}(Y, Z)) \tag{1.1}$$

is not, in general, surjective. It is always a homeomorphism onto its image. The solution is to define a class of spaces for which (1.1) is a bijection.

Definition 1.2. X is *weak Hausdorff* if for all compact Hausdorff K, and all continuous $g : K \to X$, $g(K)$ is closed.

Note that Hausdorff spaces are weak Hausdorff.

Definition 1.3. X is a *k-space* if the closed subsets are detected by maps of compact Hausdorffs into X. That is to say, $C \subseteq X$ is closed if and only if, for every compact K, $g^{-1}(C)$ is closed for every map $g : K \to X$.

Definition 1.4. X is *compactly generated* if it is a weak Hausdorff k-space.

Remark 1.5. If X is weak Hausdorff, then X is compactly generated if and only if it has the topology of the union of its compact subspaces.

Let wHaus be the category of weak Hausdorff spaces, and CG be the category of compactly generated spaces. There are adjoint pairs (wH, forget), (forget, k):

$$\mathrm{CG} \underset{k}{\overset{\text{forget}}{\rightleftarrows}} \mathrm{wHaus} \underset{wH}{\overset{\text{forget}}{\rightleftarrows}} \mathrm{Top}\,.$$

The functor k is k-ification. Given a space X, $k(X)$ is the same space, but with the closed sets changed to be those detected by maps of compacts in. The functor wH is weak Hausdorffification. It is the minimal quotient of X which is weak Hausdorff.

Proposition 1.6. The category CG is complete and cocomplete.

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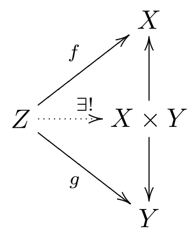
Proof. Let $X \in \mathrm{CG}^I$ be a diagram. Then easy arguments using adjoint functors show that $\varinjlim X$ and $\varprojlim X$ can be computed in CG from the colimit and limit computed in Top as follows:

$$\varinjlim^{CG} X = wH(\varinjlim^{Top} X)$$
$$\varprojlim^{CG} X = k(\varprojlim^{Top} X)$$

□

2. Products and mapping spaces

The product topology satisfies a universal property in the category Top. For spaces X and Y, giving a map $Z \to X \times Y$ is the same as giving maps $Z \to X$ and $Z \to Y$:



Since the product of weak Hausdorff spaces is weak Hausdorff, the product satisfies the same universal property in wHaus. However, the product of compactly generated spaces is not necessarily compactly generated. We do have the following lemma.

Lemma 2.1. If X is compactly generated and Y is locally compact, then $X \times Y$ is compactly generated.

We solve this problem by k-ifying: we define $X \times_{CG} Y := k(X \times Y)$.

Lemma 2.2. $X \times_{CG} Y$ satisfies the universal property in the category CG.

The mapping space may also not be compactly generated. We fix this by defining $\underline{\mathrm{Map}}_{CG}(X, Y) := k\underline{\mathrm{Map}}(X, Y)$. We then have the following theorem.

Theorem 2.3. For $X, Y, Z \in \mathrm{CG}$, the natural map

$$\underline{\mathrm{Map}}_{CG}(X \times_{CG} Y, Z) \to \underline{\mathrm{Map}}_{CG}(X, \underline{\mathrm{Map}}_{CG}(Y, Z))$$

is a homeomorphism.

From now on, we redefine the following notions:

$$\mathrm{Top} := CG$$
$$\underline{\mathrm{Map}} := \underline{\mathrm{Map}}_{CG}$$
$$- \times - := - \times_{CG} -$$

Remark 2.4. One could have (and some people do) work in the category of Hausdorff k-spaces instead of the category of weak Hausdorff k-spaces, using Hausdorffification instead of weak Hausdorffification. There are cosmetic differences which make weak Hausdorff perhaps a preferable hypothesis. Firstly, if A is a closed subset of a weakly Hausdorff k-space X, the quotient X/A, with the usual quotient topology is still a weak Hausdorff k space. By contrast, there exists examples of

Hausdorff k-spaces X with closed subsets A such that the quotient X/A is not Hausdorff. (Of course, you could apply Hausdorffification, but it is somehow appealing to think you should not need to do this to closed quotients). The second reason one might prefer the notion of weak Hausdorff in the context of k-spaces is that a general topological space is Hausdorff if and only if the diagonal $\Delta X \subset X \times X$ is closed, whereas a k-space is weak Hausdorff if and only if the diagonal $\Delta X \subset X \times_{CG} X$ is closed.

Theorem 2.3 gives an easy proof of the following proposition.

Proposition 2.5. There is a bijection $\pi_0(\underline{\mathrm{Map}}(X, Y)) \cong [X, Y]$.

Here $[X, Y]$ denotes homotopy classes of maps from X to Y.