HOMEWORK 13

DUE: 5/11/10

1) Write $H^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]$ with $|x| = 1$. Use the properties of Steenrod operations to deduce the formula

$$\mathrm{Sq}^i(x^k) = \binom{k}{i} x^{k+i}.$$

2) Use Steenrod operations to deduce that if $\alpha : S^{4k-1} \to S^{2k}$ has Hopf invariant 1, then the suspensions

$$\Sigma^n \alpha : S^{4k-1+n} \to S^{2k+n}$$

are all not null homotopic.

3) Argue that $BSO(n)$ classifies *oriented* real vector bundles of rank n, and that for X a CW complex, a rank n vector bundle is orientable if and only if the classification map

$$X \to BO(n)$$

factors through the map $BSO(n) \to BO(n)$ (induced by the inclusion $SO(n) \hookrightarrow O(n)$). In fact the short exact sequence

$$SO(n) \to O(n) \to \mathbb{Z}/2$$

induces a fiber sequence

$$BSO(n) \to BO(n) \to B\mathbb{Z}/2.$$

Deduce that a vector bundle ξ is orientable if and only if $w_1(\xi) = 0$. (Hint: I found it useful to analyze the map in cohomology $BO(n) \to B\mathbb{Z}/2$ by analyzing the composite $B\mathbb{Z}/2^n \to BO(n) \to B\mathbb{Z}/2$ in cohomology.)

4) [Milnor and Stasheff, problem 8-A] Prove the Wu formula:

$$\mathrm{Sq}^k(w_m) = \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i} w_{m+i}$$

inductively as follows: the formula is true for line bundles by problem 1, and if the formula is true for a rank n vector bundle ξ, then it is true for $\xi \oplus L$ where L is a line bundle. Therefore, by the splitting principle, the formula holds for bundles of rank $n+1$.