HOMEWORK 7

DUE DATE: TUESDAY, MARCH 30 (AFTER SPRING VACATION)

1. Argue that there exists a map $\alpha : S^2 \to S^2 \vee S^1$ so that the inclusion

$$S^1 \hookrightarrow (S^2 \vee S^1) \cup_\alpha D^3$$

induces an isomorphism on π_1 and $\widetilde{H}_*$, but is not a homotopy equivalence.

2. (Hatcher) Consider the equivalence relation $\sim_w$ generated by weak homotopy equivalence: $X \sim_w Y$ if there are spaces $X = X_1, X_2, \ldots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \sim_w Y$ iff X and Y have a common CW-approximation.

3. Show that $p : E \to B$ is a principle G-bundle, and $f : X \to B$ is a map, then the pullback $f^*E = E \times_B X \to X$ is a principle G-bundle. Show that if $g : Y \to X$ is another map, then there is an isomorphism of G-bundles

$$g^* f^* E \cong (f \circ g)^* E.$$

4. Suppose that X is a pointed, connected CW complex. A trivialization of a principle G-bundle

$$p : E \to X$$

is a section $s : X \to E$, satisfying $ps = Id_X$. An isomorphism of trivialized bundles is a bundle isomorphism which preserves the trivialization.

(a) Argue that a trivialization is the same thing as an isomorphism with the trivial G-bundle.
(b) Let $EG \to BG$ be the universal G-bundle. Argue that there is an isomorphism

$$[X, EG]_* \cong \{\text{isomorphism classes of trivialized } G\text{-bundles}\}.$$

(c) Show that any two trivialized bundles are isomorphic. Conclude that EG is contractible, and that ΩBG is weakly equivalent to G.

Note: you may assume that G is discrete. Then $EG \to BG$ is a covering space, and you can use the existence and uniqueness theorems for lifts to a covering space. But I think the whole problem works for G a more general topological group. Also, you might find it easier to establish EG is contractible some other way. Then (b) follows from (c), since then you would know both functors in question take constant, singleton values!