11 - Chromatic tower

Note Title 10/28/2008

$E(n)$ + $K(n)$

We have seen

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{Cx oriented}\\ \text{ring spectra}\end{array}\right\} \arrow[r] & \{\text{FGL's}\} \arrow[l, bend left, dashed, "\text{LEFT}"]
\end{tikzcd}$$

Landweber exact functor theorem

Suppose that A is a graded commutative BP_*-alg

$$(BP_* \longrightarrow A) \longleftrightarrow (\text{p-typical FGL}/A)$$

$$(p, v_1, v_2, ---) \qquad (p, a_1, a_2, ---)$$

and that $(p, a_1, a_2, ---)$ is a regular sequence

$$(*) \quad \left(\forall n \quad a_n : A/(p, ---, a_{n-1}) \hookrightarrow A/(p, --- a_{n-1}) \right)$$

Then $\exists$ ring spectrum $BP \to E$ such that

$$E_*(X) \cong A \otimes_{BP_*} BP_*(X)$$

$$(F_E \text{ is classified by } BP_* \longrightarrow E_* = A.)$$

Key idea!

$$(\star) \iff A_* \otimes_{BP_*} - : \mathrm{Comod}_{BP_*BP} \longrightarrow Ab \text{ is exact}$$

[**Note** $A_* \otimes_{BP_*} - : \mathrm{Mod}_{BP_*} \longrightarrow Ab$ may not be exact]

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e.g. $A = \mathbb{Z}_{(p)}[v_1, v_2, ---, v_n, v_n^{-1}]$

$$A/I_m = \begin{cases} \mathbb{F}_p[v_m, v_{m+1}, ---, v_n, v_n^{-1}] & m \leq n \\ 0 & m > n \end{cases}$$

$\longrightarrow$ $E(n)$ homotopy commutative ring spectrum

$E(n)_* = \mathbb{Z}_{(p)}[v_1, ---, v_n^{\pm 1}]$

Johnson-Wilson theory

Thm (Robinson et al ---, Goerss-Hopkins)

$E(n)$ is E_∞

EKMM

$R =$ commutative S-alg

$x \in \pi_d R$ not a zero divisor.

$$S^d \xrightarrow{x} R \qquad S\text{-modules}$$

$$\Updownarrow$$

$$\Sigma^d R \xrightarrow{x} R \qquad R\text{-modules}$$

Defn $\Sigma^d R \longrightarrow R \longrightarrow R/x$ $\Big|$ $\pi_* R/x \cong R_*/(x)$

Thm If $\pi_{2d+1} R/x = 0 \Rightarrow R/x$ is a homotopy unital ring spectrum

$\pi_{3d+3} R/x = 0 \Rightarrow R/x$ is a htpy assoc ring spectrum

More generally,

If $\pi_* R$ is concentrated in even degrees

$(x_1, x_2, \ldots) \in \pi_* R$ regular sequence

htpy assoc
ring spec.
✓

Can induct for $\Sigma^{d_{n+1}} R/(x_1, x_2, \ldots, x_n) \xrightarrow{x_{n+1}} R/(x_1, \ldots x_n) \longrightarrow R/(x_1 \ldots x_{n+1})$

Moreover can invert $y \in \pi_d R$

$$R[y^{-1}] := \mathrm{Tel}\left(R \xrightarrow[y]{} \Sigma^{-d} R \xrightarrow[y]{} \Sigma^{-2d} R \xrightarrow[y]{} \cdots \right)$$

Fact: MU E_∞-ring

$\Rightarrow$ $MU_{(p)}$ E_∞-ring

BP is a regular quotient of $MU_{(p)}$

$\Rightarrow$ $\exists$ regular ideal J of $(MU_{(p)})_*$

s.t. $(MU_{(p)})_*/J \cong \mathbb{F}_p[v_n]$

$\Rightarrow$ Can form htpy assoc ring spectrum $K(n)$

s.t. $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$

"Morava K-thy"

Thm (Robinson)

$K(n)$ is A_∞

Strickland $K(2)$ at $p=2$ is **NOT** htpy commutative!

E.g. $n=1$

Thm (Adams)

$$K_{(p)} \simeq \bigvee_{i=0}^{p-1} \Sigma^{2i} E(1)$$

$$E(1) \hookrightarrow K_{(p)}$$

$$\mathbb{Z}_{(p)}[v_1^{\pm 1}] \qquad \mathbb{Z}_{(p)}[\beta^{\pm 1}]$$

$$v_1 \longmapsto \beta^{p-1}$$

$$K_{(p)}/p \simeq K_{(p)} \wedge M(p) \simeq \bigvee_{i=1}^{p-1} \Sigma^{2i} K(1)$$

$\langle E \rangle$ = class of coh thrs E' s.t.

$$L_E = L_{E'}$$

i.e. $\{E\text{-w.e.'s}\} = \{E'\text{-w.e.'s}\}$

Note: $\langle E \vee E \rangle = \langle E \rangle$

So $\langle K_{(p)} \rangle = \langle E(1) \rangle$

$$X_{M(p)} := X_p$$

$$X_{K(1)} = (X_{E(1)})_p$$

Localization wrt periodic cohomology

Example $M(p)_K = M(p)_{E(1)} = M(p)_{K(1)}$

We will show that $M(p)_K$ has non-trivial negative htpy gps.

p odd

$$\begin{array}{ccccccc} \cdots & \pi_{2p-4}S & & \pi_{2p-3}S & & \pi_{2p-2}S & \cdots \\ & 0 & & \mathbb{Z}/p\{\alpha_1\} & & 0 & \end{array}$$

Lem

htpy unital

$M(p)$ is a n ring spectrum

(not true for $p=2$!)

$$S^0 \xrightarrow[p]{} S^0 \longrightarrow M(p) \longrightarrow S^1$$

Argument

$$\begin{tikzcd} M(p)\wedge S \arrow[r, "\begin{array}{c} M(p)\wedge p \\ \shortparallel \\ p\cdot Id \end{array}"'] & M(p)\wedge S \arrow[r] & M(p)\wedge M(p) \arrow[r] \arrow[l, bend left, dashed, "\mu\ ?"] & \Sigma M(p) \end{tikzcd}$$

$\exists\ \mu \iff M(p)\wedge p \simeq 0$

$$\begin{tikzcd}
{[S^1, M(p)]} \arrow[r] \arrow[d, phantom, "\shortparallel" description] & {[M(p), M(p)]} \arrow[r] \arrow[d, phantom, "\shortparallel" description] & {[S, M(p)]} \arrow[r] \arrow[d, phantom, "\shortparallel" description] & {[S, M(p)]} \arrow[d, phantom, "\shortparallel" description] \\
0 & \mathbb{Z}/p \arrow[r, equal, "\sim"] & \mathbb{Z}/p & \mathbb{Z}/p \arrow[l, "\cdot p"'] \arrow[d] \\
 & & & 0 \arrow[ul]
\end{tikzcd}$$

$\mathbb{Z}/2\{\eta\}$

Note: $[M(2), M(2)] = \mathbb{Z}/4$!

$\alpha_1 \in \pi_{2p-3} S$

$$\begin{tikzcd}
 & \Sigma^{-1} M(p) \arrow[d] \\
S^{2p-3} \arrow[ur, dashed, "\tilde{v}"] \arrow[r, "\alpha_1"'] & S^0 \arrow[d, "p"] \\
 & S^0
\end{tikzcd}$$

get $v : \Sigma^{2p-2} M(p) \longrightarrow M(p)$

Lem:

$$\begin{tikzcd}
E(1)_* M(p) \arrow[r, "v"] \arrow[d, phantom, "\cong" rotate=90] & E(1)_* M(p) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{F}_p[v_1^{\pm 1}] \arrow[r, "\cdot v_1"'] & \mathbb{F}_p[v_1^{\pm 1}]
\end{tikzcd}$$

(pf) Suffices to show

$$\begin{tikzcd}
\pi_{2p-2} M(p) \arrow[r, "u \wedge 1"] & \pi_{2p-2} E(1) \wedge M(p) \\
\tilde{v} \arrow[r, mapsto] & v_1
\end{tikzcd}$$

Use ANSS:

$$\begin{aligned}
BP_* E(1) \wedge X &\cong E(1)_*(BP \wedge X) \\
&\underset{\text{LEFT}}{\cong} E(1)_* \otimes_{BP_*} BP_* BP \wedge X \\
&\underset{\text{use conjugate}}{\cong} \underbrace{BP_* BP \otimes_{BP_*} E(1)_* X}_{\text{extended comodule}}
\end{aligned}$$

So

$$\begin{aligned}
&\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_* E(1)_* X) \\
\cong\ &\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*BP \otimes_{BP_*} E(1)_* X)
\end{aligned}$$

$$\cong \begin{cases} \mathrm{Hom}_{BP_*}(BP_*, E(1)_*X) = E(1)_*X & s=0 \\ 0 & s>0 \end{cases}$$

$$\begin{tikzcd}
BP_* \arrow[r, "\cdot p"'] & BP_* \arrow[r] & BP_*M(p)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ext}_s(BP_*, BP_*\Sigma^{-1}M(p)) \arrow[rr, "\partial"'] \arrow[ddd] & & \mathrm{Ext}(BP_*, BP_*) \\
 & & \varphi \\
 & v_1 \arrow[r, mapsto] \arrow[dd, mapsto] & \alpha_1 \\
\mathrm{Ext}_*(BP_*, E(1)_*\Sigma^{-1}M(p)) & & \\
 & v_1 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & v_1 \arrow[r, mapsto] \arrow[d, "d"] & v_1 \\
t_1 \arrow[r] & p t_1 &
\end{tikzcd}$$

$\alpha_1 \longleftrightarrow [t_1]$

$//$

Corollary • $v^i \neq 0 \quad \forall i$

• v is an $E(1)$-equivalence

$\exists \quad v^{-1} : \Sigma^{-(p-1)} M(p)_{E(1)} \longrightarrow M(p)_{E(1)}$

$$\begin{tikzcd}
S^{2i(p-1)} \arrow[r] & \Sigma^{2i(p-1)} M(p)_{E(1)} \arrow[r, "v^i"'] & M(p)_{E(1)}
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{2i(p-1)} M(p)_{E(1)} \arrow[r] & \pi_{2i(p-1)} E(1) \wedge M(p) \arrow[d, phantom, sloped, "="] \\
 & \mathbb{F}_p[v_1^{\pm 1}] \\
v^i \arrow[uu, phantom, sloped, "\in"] \arrow[r, mapsto] & v_1^i \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

i negative $\Rightarrow$ non-trivial negative htpy grps!

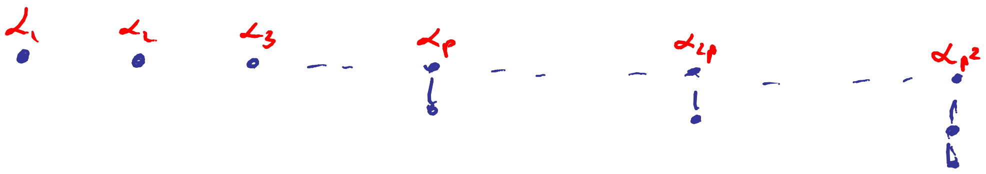
<u>periodic</u> family

<u>Moreover</u> get a periodic family in $\pi_* S_{E(1)}$ as follows

$$\begin{tikzcd}
S^{2i(p-1)-1} \arrow[r] \arrow[rrr, bend right, "\alpha_i"'] & \Sigma^{2i(p-1)-1} M(p)_{E(1)} \arrow[r, "v^i"'] & \Sigma^{-1} M(p)_{E(1)} \arrow[r] & S_{E(1)}
\end{tikzcd}$$

Note: $i = 1 \Rightarrow \alpha_1$

Moreover can remove $\Theta_{E(1)}$ if $i > 0$



Constructed elts of order p in $\operatorname{Im} J$!

"v_1-periodic"

Vast generalization:

Lemma $\{E(n)\text{-equivalences}\} \subseteq \{E(n-1)\text{-equivalences}\}$

Basic idea: use Landweber filtration thm:

$M =$ f.p BP_*-module, BP_*BP-comodule

$\Rightarrow$ M admits a finite filtration whose quotients are $\oplus BP_*/I_m$'s

$$E(n)_* \otimes_{BP_*} BP_*/I_m = \begin{cases} 0, & m > n \\ \underline{BP_*/I_m [v_n^{-1}]}, & m \le n \\ (v_{n+1}, v_{n+2}, \ldots) & \end{cases}$$

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argue that

$$\langle E(n) \rangle = \langle BP[v_n^{-1}] \rangle$$

and $BP[v_n^{-1}]$-equivalence $\Rightarrow$ $BP[v_{n-1}^{-1}]$-equivalence

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in fact $\langle E(n) \rangle = \langle K(0) \vee K(1) \vee \cdots \vee K(n) \rangle$

$$K(0) = E(0) = HQ$$

$$\mathbb{Z}_{(p)}[v_0^{-1}] = \mathbb{Q}$$

X = spectrum:

chromatic tower

$$\cdots \to X_{E(n)} \to X_{E(n-1)} \to \cdots \to X_{E(0)}$$

X is said to be *harmonic* if

$$X_{(p)} \simeq \underleftarrow{\mathrm{holim}}\, X_{E(n)}$$

- ∃ non-harmonic spectra
- *Chromatic convergence thm*: Suspension spectra are harmonic.

$$\Longrightarrow \quad S_{(p)} \simeq \underleftarrow{\mathrm{holim}}\, S_{E(n)}$$

Chromatic Spectral Sequence X harmonic

$$\begin{tikzcd}
\cdots \arrow[r] & X_{E(2)} \arrow[r] & X_{E(1)} \arrow[r] & X_{E(0)} \arrow[d, equal] \\
 & M_2X \arrow[u] & M_1X \arrow[u] & M_0X \arrow[d, equal] \\
 & & & X_{\mathbb{Q}}
\end{tikzcd}$$

Get

$$E_1^{s,t} = \pi_t M_s X \Longrightarrow \pi_t X_{(p)}$$

Key point: layers are computable

associated filtration on $\pi_* X$
is called chromatic filtration.