13 - Morava Change of Rings

Note Title 11/12/2008

Chromatic layers $\longleftrightarrow \mathrm{Ext}_{BP_*BP}\left(BP_*/(p^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]\right)$

Compute this?

Inductive computation of $\mathrm{Ext}_{BP_*BP}\left(BP_*/(p, \dots, v_m, v_{m+1}^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]\right)$

$$\left(BP_*/I_m\right)/(v_m^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]$$

$$\parallel$$

$$\operatorname{colim}_K \left(BP_*/I_m\right)/(v_m^K, v_{m+1}^\infty, \dots)[v_n^{-1}]$$

Get a BSS

$$\mathrm{Ext}\left(\left(BP_*/I_{m+1}\right)/(v_{m+1}^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]\right)\left[\frac{1}{v_m^K}\right]$$

$$\Longrightarrow \mathrm{Ext}\left(\left(BP_*/I_m\right)/(v_m^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}]\right)$$

diff'ls $\qquad x \in C^s\left(BP_*/(v_{m+1}^{i_{m+1}}, \dots, v_{n-1}^{i_{n-1}})[v_n^{-1}]\right)$

$$\left[\frac{x}{v_{m+1}^{i_{m+1}} \cdots v_{n-1}^{i_{n-1}}}\right] \in \mathrm{Ext}^s_{BP_*}\left(\left(\frac{BP_*}{I_{m+1}}\right)\Big/(v_{m+1}^{\infty}, \dots, v_{n-1}^{\infty})[v_n^{-1}]\right)$$

$$d(x) \equiv 0 \quad (p, v_1, \dots, v_m, v_{m+1}^{i_{m+1}}, \dots, v_{n-1}^{i_{n-1}})$$

In BSS

$$d\left[\frac{x}{v_m^{k+r} v_{m+1}^{i_{m+1}} \cdots v_{n-1}^{i_{n-1}}}\right] = \frac{y}{v_m^{k} v_{m+1}^{i_{m+1}} \cdots v_{n-1}^{i_{n-1}}}$$

$$\Longleftrightarrow \qquad dx = v_m^r y \quad (p, v_1, \dots, v_{m-1}, v_m^{r+k}, v_{m+1}^{i_{m+1}}, \dots, v_{n-1}^{i_{n-1}})!$$

$$\mathrm{Ext}_{BP_*}\big(\underbrace{BP_*/I_n[v_n^{-1}]}_{K(n)}\big) \underset{v_{n-1}\ \mathrm{BSS}}{\Longrightarrow} \cdots\cdots \underset{v_0\ \mathrm{BSS}}{\Longrightarrow} \mathrm{Ext}\left(BP_*/(p^{\infty}, \dots, v_{n-1}^{\infty})[v_n^{-1}]\right)$$

Morava Change of rings thm

$\Sigma(n) =$ certain Hopf algebroid$/K(n)_*$ "Morava stabilizer algebra"

$\mathbb{S}_n =$ certain pro-p-gp gp "Morava stabilizer gp"

$S(n) = \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p \qquad v_n \longmapsto 1$

Change of Rings:

$$\mathrm{Ext}_{BP_*BP}(K(n)_*) \underset{(1)}{\cong} \mathrm{Ext}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)[v_n^{\pm 1}]$$

$$\mathrm{Ext}_{BP_*BP}(K(n)_*) \overset{(2)}{\cong} H_c^*(S_n, \mathbb{F}_p)[v_n^{\pm 1}]$$

$$\mathrm{Ext}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)[v_n^{\pm 1}] = \mathrm{Ext}_{\Sigma(n)}(K(n)_*, K(n)_*)$$

$$\Sigma(n) = E(n)_*E(n) / (p, v_1, \dots, v_{n-1}) = K(n)_*E(n)$$

So (1) $\longleftrightarrow$ algebraic telescope conj:

$$\mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \cong \mathrm{Ext}_{BP_*BP}(BP_*, BP_*/I_n[v_n^{-1}])$$

$$\mathrm{Ext}_{E(n)_*E(n)}(E(n)_*, E(n)_*/I_n) \cong \mathrm{Ext}_{E(n)_*E(n)/I_n}(E(n)_*/I_n, E(n)_*/I_n)$$

ANSS for $v_n^{-1} V(n-1)$

ANSS for $V(n-1)_{E(n)}$

We will sketch

$$\mathrm{Ext}_{BP_*BP}(BP_*/I_n[v_n^{-1}]) \underset{(2)}{\cong} H^*(S_n, \mathbb{F}_p)[v_n^{-1}] \underset{(3)}{\cong} \mathrm{Ext}_{S(n)}(\mathbb{F}_p)[v_n^{-1}]$$

Idea:

behind (2)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\sim", "\text{equiv of ab cats}"'] \arrow[dr, "F"'] & & \mathcal{B} \arrow[dl, "G"] \\ & \mathrm{Ab} & \end{tikzcd}$$

$$\Rightarrow \quad R^i F \cong R^i G$$

[Warning: finiteness assumptions omitted]

$\mathcal{A}$ $\left\{ \begin{array}{l} BP_*BP\text{-comodules } M \\ \text{annihilated by } (p, \ldots, v_{n-1}) \\ \quad v_n \text{ acts invertibly} \end{array} \right\}$ $\qquad M$

$\simeq \updownarrow \qquad \mathrm{Spec}(BP_*),\ \mathrm{Spec}(BP_*BP)$ $\qquad \updownarrow \qquad$ modules $\updownarrow$ quasi-coh. sheaves

$\qquad\qquad$ " p-typical FGL's / strict isos

$\mathcal{B}$ $\left\{ \begin{array}{l} R = \text{comm ring} \qquad \underline{\text{Functors}}\text{: natural in } R \\ \qquad \mathfrak{G}^n_R \longrightarrow M_{(R,F)} \\ \text{Ob's: } F = p\text{-typical FGL}/R \\ \qquad \text{s.t. } [p]_F = v_n x^{p^n} +_F \cdots \qquad \text{"Ht } F = n\text{"} \\ \qquad\qquad v_n \text{ invertible} \\ \text{Morph's = strict isos} \end{array} \right\}$ $\leftarrow$ secretly "coherent sheaf on stack"

$BP_* \xrightarrow[F]{} R$

$M_{(R,F)} := R \otimes_{BP_*} M$

$\simeq \updownarrow$ (*) $\qquad (\mathcal{M}_{FG} \otimes \mathbb{F}_p)_{Ht\, n}$

$$\mathcal{C}\left\{ S_n \rtimes Gal\text{-modules}/\overline{\mathbb{F}}_p \qquad M_0 \right\} \qquad M_0 = M_{(\overline{\mathbb{F}}_p, F_n)}$$

F_n = p-typical formal gp $/\overline{\mathbb{F}}_p$

$$[p]_{F_n} = x^{p^n}$$

$$\mathbb{S}_n = \mathrm{Aut}^{\text{strict}}(F_n)$$

$$Gal = Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

(*) follows from

(1) $\underline{\underline{\text{Thm}}}$ FG's / separably closed flds k char p classified by Height

$\Rightarrow$ Groupoid $\mathcal{F}^n_k \simeq \overset{F_n}{*} \circlearrowleft \mathrm{Aut}(F)$

1 object

(2) Sheaf is determined by "Stalks at $\overline{\mathbb{F}}_p$ points"

$$M \longmapsto M \otimes_{BP_*} \overline{\mathbb{F}}_p$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, leftrightarrow, "\sim"] \arrow[dr] & \mathcal{B} \arrow[r, leftrightarrow, "\sim"] & \mathcal{C} \arrow[dl] \\
& \mathrm{Ab} &
\end{tikzcd}$$

$$\mathrm{Hom}_{BP_*BP}(BP_*, M) \cong \left(M \otimes_{BP_*} \overline{\mathbb{F}}_p\right)^{S_n \rtimes Gal} [v_n^{-1}]$$

Get

$$\mathrm{Ext}_{BP_*BP}(BP_*, M) \cong H^*_c\left(S_n, M \otimes_{BP_*} \overline{\mathbb{F}}_p\right)^{Gal}$$

<u>In particular</u>:

$$\mathrm{Ext}_{BP_*BP}\left(BP_*, BP_*/I_n [v_n^{-1}]\right) \cong H^*_c(S_n, \overline{\mathbb{F}}_p)^{Gal} [v_n^{-1}]$$

trivial S_n-mod

$$\cong H^*(S_n, \mathbb{F}_p)[v_n^{-1}]$$

(3) $$H^*(S_n, \mathbb{F}_p) \cong \mathrm{Ext}_{S(n)}(\mathbb{F}_p)$$

$$H_c^*(S_n, \bar{\mathbb{F}}_p) \cong \mathrm{Ext}_{S(n)}(\bar{\mathbb{F}}_p) \quad + \text{ "Galois descent"}$$

$$\Sigma_c(n) = K(n)_* E(n)$$

$$E(n)_* E(n) = E(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} E(n)_*$$

Landweber exactness

$$= \frac{E(n)_*[t_1, t_2, \dots]}{(\eta_R(v_{n+1}), \eta_R(v_{n+2}), \dots)}$$

$$K(n)_* E(n) = \frac{K(n)_*[t_1, \dots]}{(\eta_R(v_{n+1}), \dots)}$$

Lem:

$$\Sigma_c(n) = K(n)_* E(n) \cong K(n)_* \otimes_{BP_*} BP_* BP \otimes_{BP_*} K(n)_*$$

idea:

$$\mathrm{Ring}(K(n)_* E(n), R)$$

$$\updownarrow$$

$$F \xrightarrow[\text{strict iso}]{} F' \qquad F \text{ has ht } n \implies F' \text{ ht } n$$

Using inductive η_R formula

$$K(n)_*E(n) \cong \frac{\mathbb{F}_p[v_n^{\pm 1}][t_1, t_2, \ldots]}{\left(v_n t_k^{p^n} = v_n^{p^k} t_k\right)_{k \geq 1}}$$

$$\cong S(n) \otimes_{K(n)_*} \mathbb{F}_p$$

(ungraded isomorphism)

$$K(n)_* \longrightarrow \mathbb{F}_p$$
$$v_n \longmapsto 1$$

$$S(n) \cong \frac{\mathbb{F}_p[t_1, t_2, \ldots]}{\left(t_k^{p^n} = t_k\right)_{k \geq 1}}$$

(actually $\mathbb{Z}/2(p^n-1)$-graded)

<u>Note</u>

$$S_n = \mathrm{Aut}^{\mathrm{strict}}(F_n) \cong \mathrm{Ring}\left(S(n), \overline{\mathbb{F}}_p\right)$$

$$\cong \mathrm{Ring}\left(S(n), \mathbb{F}_{p^n}\right)$$

$t_k^{p^n} = t_k \Rightarrow t_k \in \mathbb{F}_{p^n} \subset \overline{\mathbb{F}}_p$

$$\mathrm{Ext}_{S(n)\otimes\mathbb{F}_{p^n}}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}) \cong H^*_c(S_n, \mathbb{F}_{p^n})$$

<u>Compare</u> k = field G = finite

$$\mathrm{Ext}_{k[G]}(k, k) \cong H^*(G, k)$$

<u>Program</u>

$$\mathrm{Ext}_{S(n)}(\mathbb{F}_p, \mathbb{F}_p)[v_n^{-1}] \cong \mathrm{Ext}_{BP_*BP}(BP_*/I_n[v_n^{-1}])$$

use May SS

Morava change of rings

$\Downarrow$ Sequence of BSS's

$$\mathrm{Ext}_{BP_*BP}(BP_*/(p^\infty, \dots, v_{n-1}^\infty)[v_n^{-1}])$$

$\Downarrow$ ANSS

$$\pi_* M_n S$$

$\Downarrow$ CSS

$$\pi_* S$$