Note Title 9/10/2008

# $H\mathbb{F}_p$ - the classical ASS + the Steenrod Alg

$E = H\mathbb{F}_p$ $\quad p =$ prime

<u>ASS</u>

$$\mathrm{Ext}^{s,t}_{H\mathbb{F}_{p*}H\mathbb{F}_p}(H\mathbb{F}_{p*}, H\mathbb{F}_{p*}) \Longrightarrow \pi_{t-s} S^\wedge_p$$

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$$H\mathbb{F}_p^* H\mathbb{F}_p = [H\mathbb{F}_p, \Sigma^* H\mathbb{F}_p]$$
$$\simeq \overset{\text{stable}}{\mathrm{Nat}}(H\mathbb{F}_p^*(-), H\mathbb{F}_p^*(-))$$
$$= A$$

mod $p$ Steenrod algebra

$A$ is (non-commutative) $\mathbb{F}$-algebra generated by

$$\begin{cases} Sq^i, & |Sq^i| = i \qquad p = 2 \\ P^i, \beta, & |P^i| = 2i(p-1), \quad |\beta| = 1 \qquad p > 2 \end{cases}$$

<u>Subject to:</u>

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2j} Sq^{i+j-k} Sq^k \quad \text{for } 0 < i < 2j. \qquad p = 2$$

$P^a P^b = \text{-----}$
$P^a \beta P^b = \text{----}$ $\qquad p > 2$

Indecomposables $\{Sq^{2^i}\}$ $p = 2$

$\{P^{p^i}, \beta\}$ $p > 2$

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Apparent assymetry is corrected by

$$Sq^1 \longleftrightarrow \beta$$

$$Sq^{2i} \longleftrightarrow P^i$$

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$A$ is a Hopf algebra (cocommutative)

coproduct

Cartan formula

$$\psi(Sq^i) = \sum_{i_1+i_2=i} Sq^{i_1} \otimes Sq^{i_2}$$

$$\psi(\beta) = \beta \otimes 1 + 1 \otimes \beta$$

$$\psi(P^i) = \sum_{i_1 \otimes i_2} P^{i_1} \otimes P^{i_2}$$

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Since $A$ is finitely generated in each degree

$\mathbb{F}_p$ = fld

$$H\mathbb{F}_{p*} H\mathbb{F}_p \cong \mathrm{Hom}_{\mathbb{F}_p}(A, \mathbb{F}_p)$$

$$\begin{matrix} \| \\ A_* \end{matrix}$$

Adem relations $\Rightarrow$ $A$ has basis by

Admissible monomials.

$p=2$:

e.g.

$$\{ Sq^{i_1} \cdots Sq^{i_k} \}$$

$$i_j \geq 2 i_{j+1}$$

Wrt this basis:

$p=2$ $\quad \xi_k = \left( Sq^{2^{k-1}} Sq^{2^{k-2}} \cdots Sq^{2^0} \right)^*$

$p>2$ $\quad \xi_k = \left( P^{p^{k-1}} \cdots \cdots P^{p^0} \right)^*$

$$\tau_k = \left( P^{p^{k-1}} \cdots P^{p^0} \beta \right)^*$$

Thm (Milnor)

$$A_* \cong \begin{cases} \mathbb{F}_2[\xi_1, \xi_2, \ldots] & p=2 \\ \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \ldots] & p>2 \end{cases}$$

$|\xi_i| = 2^i - 1$

$|\xi_i| = 2(p^i - 1)$

$|\tau_i| = 2p^i - 1$

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$A_*$ is a commutative (but not cocommutative) Hopf algebra:

$\underline{\underline{p=2}}$ $$\psi(\xi_i) = \sum_{i_1+i_2=i} \xi_{i_1}^{2^{i_2}} \otimes \xi_{i_2}$$

[convention $\xi_0 = 1$]

$\underline{\underline{p>2}}$ $$\psi(\xi_i) = \sum_{i_1+i_2=i} \xi_{i_1}^{p^{i_2}} \otimes \xi_{i_2}$$

$$\psi(\tau_i) = \sum_{i_1+i_2=i} \xi_{i_1}^{p^{i_2}} \otimes \tau_{i_2} + \tau_i \otimes 1$$

Compare:

| $p=2$ | | $p>2$ |
|---|---|---|
| $\xi_i$ | $\longleftrightarrow$ | $\tau_i$ |
| $\xi_i^2$ | $\longleftrightarrow$ | $\xi_i$ |

Milnor Basis:

given $A_*$ monomial basis

$$\{\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_k^{e_k}\} \qquad p=2$$

$$\{\tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}\cdots\tau_k^{\varepsilon_k}\,\xi_1^{e_1}\xi_2^{e_2}\cdots\xi_\ell^{e_\ell}\} \qquad p>2$$

gives DUAL BASIS of $A$
(Milnor)

$$\{Sq(e_1, e_2, \ldots, e_k)\}$$

where $Q_i$ dual to $\tau_i$

$$\{Q_0^{\varepsilon_0}Q_1^{\varepsilon_1}\cdots Q_k^{\varepsilon_k}\,P(e_1, \ldots, e_\ell)\}$$

$$\Lambda_{\mathbb{F}_p}[Q_0, Q_1, \ldots] \underset{\text{subalgebra}}{\subset} \mathcal{A}$$

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<u>Note:</u>

$$Sq^i = Sq(i) \longleftrightarrow \xi_1^i \qquad p=2$$

$$\left.\begin{array}{l} P^i = P(i) \longleftrightarrow \xi_1^i \\ \beta = Q_0 \longleftrightarrow \tau_0 \end{array}\right\} \quad p>2$$

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"Important" sub-algebras

$$A(n) \subset A$$

$$A(n) = \begin{cases} \langle Sq^1, \ldots, Sq^{2^n} \rangle, & p=2 \\ \langle P^1, \ldots, P^{p^{n-1}}, \beta \rangle, & p>2 \end{cases}$$

$P_s^t$'s: <u>At all primes:</u>

$P_s^t$ is dual to $\xi_s^{p^t}$

$$A \text{ generated by } \begin{cases} P_s^t, & s \geq 1,\ t \geq 0,\ p=2 \\ P_s^t,\ s \geq 1,\ t \geq 0 \text{ and } & , \ p \text{ odd} \\ Q_i,\ i \geq 0 & \end{cases}$$

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| $p=2$ | | $p$ odd |
|---|---|---|
| $P_s^0$ | $\longleftrightarrow$ | $Q_s$ |
| $P_s^{t+1}$ | $\longleftrightarrow$ | $P_s^t$ |

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$$A(n) = \begin{cases} \langle P_s^t \mid s+t \leq n+1 \rangle, & p=2 \\ \langle P_s^t, Q_i \mid \begin{matrix} s+t \leq n \\ i \leq n \end{matrix} \rangle, & p>2 \end{cases}$$

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## Significance of $P_s^t$'s

$$I \to A \xrightarrow[\varepsilon]{} \mathbb{F}_p$$

$$I/I^2 = Q(A)$$

$$= \mathbb{F}_2\{Sq^1, Sq^2, Sq^4, \cdots\} \qquad p=2$$

$$P_1^0 \quad P_1^1 \quad P_1^2$$

$$I^2/I^3 = \mathbb{F}_2\{P_1^i P_1^j \mid i>j\} \oplus \mathbb{F}_2\{P_2^0, P_2^1, \ldots\}$$

$$\left[\begin{array}{l} \psi(\xi_2^{2^t}) = \xi_1^{2^{t+1}} \otimes \xi_1^{2^t} + \cdots \\ \Longrightarrow P_1^{t+1} P_1^t = P_2^t + \cdots \end{array}\right]$$

$\vdots$

$E^0A$ is generated by $P_s^t$'s

**p odd**

$E^0A$ is generated by $P_s^t$'s + $Q_i$'s

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$E^0A$ is a primitively generated Hopf algebra$/\mathbb{F}_p$

$$PA \to \bar{A} \xrightarrow[\psi]{} \bar{A} \otimes \bar{A}$$

primitives of $A$

$$k \xrightarrow[\eta]{} A \to \bar{A}$$

$x \in PA \Rightarrow \psi(x) = x \otimes 1 + 1 \otimes x$

Dual to indecomposables notion

$$I \otimes I \xrightarrow[\mu]{} I \to QA$$

$E^0 A$ is primitively generated

Hopf algebra / $\mathbb{F}_p$

$$\psi(P_s^t) = P_s^t \otimes 1 + 1 \otimes P_s^t + \cdots$$

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Thm (Milnor–Moore)

$$E^0 A = V(\underbrace{PE^0 A}_{\text{RESTRICTED LIE ALGEBRA}})$$

$V$: universal enveloping alg of restricted lie alg.

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Restricted Lie alg

$$[\,,\,] : L \times L \longrightarrow L$$

$$\xi : L \longrightarrow L$$

$$\left[ V(\xi) = T(L) \Big/ \begin{matrix} xy - yx = [x,y] \\ x^p = \xi(x) \end{matrix} \right]$$

$\Rightarrow$ $E^0 A$ is **additively** given by

$p=2$

$$\Lambda_{\mathbb{F}_2}[P_s^t \mid s \geq 1,\ t \geq 0]$$

$p$ **odd**

$$Tr_{\mathbb{F}_p}\left[P_s^t \;\middle|\; \begin{array}{c} s \geq 1 \\ t \geq 0 \end{array}\right] \otimes \Lambda_{\mathbb{F}_p}[Q_i \mid i \geq 0]$$

truncated polynomial ring

$$Tr_k[x] = k[x]/x^p$$

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$E^0 A(n)$ is **additively** given by

$$\Lambda_{\mathbb{F}_2}\left[P_s^t \;\middle|\; \begin{array}{c} s \geq 1,\ t \geq 0 \\ s+t \leq n+1 \end{array}\right] \qquad p = 2$$

$$Tr_{\mathbb{F}_p}\left[P_s^t \;\middle|\; \begin{array}{c} s \geq 1 \\ t \geq 0 \\ s+t \leq n \end{array}\right] \otimes \Lambda_{\mathbb{F}_p}\left[Q_i \;\middle|\; \begin{array}{c} i \geq 0 \\ i \leq n \end{array}\right]$$

$p$ odd

$$A(n) \hookrightarrow A$$

$$A_* \twoheadrightarrow A(n)_*$$

$\underline{p=2}$

$$\mathbb{F}_2[\xi_1, \xi_2, \ldots] \twoheadrightarrow \mathbb{F}_2[\xi_1, \ldots, \xi_{n+1}] \big/ (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \ldots, \xi_{n+1}^2)$$

$\underline{\underline{p \text{ odd}}}$

$$\mathbb{F}_p[\xi_1, \xi_2 \ldots] \otimes \Lambda[\tau_0, \tau_1 \ldots] \twoheadrightarrow \mathbb{F}_p[\xi_1, \ldots, \xi_n] \big/ (\xi_1^{p^n}, \ldots, \xi_n^p) \otimes \Lambda[\tau_0, \ldots, \tau_n]$$

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Relationship between

$\mathrm{Ext}_{A(n)_*}$ + $\mathrm{Ext}_{A_*}$ ....

Leads us to discussi

**Aside:** $/\!\!/$ and $\square$

$B$ = Hopf algebra $/k$ ← std

$C \subseteq B$ sub-Hopf algebra

$$B/\!\!/C := B \otimes_C k$$

$C \to k$ (augmentation)

"exact sequence"

$$C \longrightarrow B \longrightarrow B/\!\!/C$$

**Warning:** $B/\!\!/C$ might not inherit Hopf alg structure

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$B^* =$ dual of $B$

$$B^* \twoheadrightarrow C^*$$

**Problem:** identify $(B/\!\!/C)^*$

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$M$ = right $B$-module

$N$ = left $B$-mod

coequalizer

$$\left\{ \begin{array}{c} M \otimes_B N \\ \uparrow \\ M \otimes N \\ {\scriptstyle 1\otimes ac}\uparrow\uparrow{\scriptstyle ac\otimes 1} \\ M \otimes B \otimes N \end{array} \right\}$$

Duality:

$M$ = right $B$-comodule

$N$ = left $B$-comodule

$(B/\!/C)^* \cong B^* \square_{C^*} k$

$$M \underset{B}{\square} N \longrightarrow M \otimes N \underset{1\otimes\psi}{\overset{\psi\otimes 1}{\rightrightarrows}} M \otimes B \otimes N$$

equalizer

Universal Property:

$M \in \mathrm{Mod}_C$

$N = \mathrm{Mod}_B$

$$\mathrm{Mod}_B(B \otimes_C M, N) \cong \mathrm{Mod}_C(M, N)$$

$\Longrightarrow$ Change of rings spectral sequence (Grothendieck Spectral sequence)

$$\mathrm{Ext}_B^*(\mathrm{Tor}_*^C(B, M), N) \Rightarrow \mathrm{Ext}_C^*(M, N)$$

So: If $B$ is $C$-flat, then (e.g. $C$-free)

(Change of rings isomorphism)

$$\mathrm{Ext}_B^*(B \otimes_C M, N) \cong \mathrm{Ext}_C^*(M, N)$$

**Duality**: Universal property for $B^* \Box_{C^*} -$:

$M \in CoMod_{B^*}$
$N \in CoMod_{C^*}$

$$CoMod_{B^*}(M, B^* \Box_{C^*} N) \cong CoMod_{C^*}(M, N)$$

$$CoTor^*_{B^*}(M, N) := R^*(M \Box_{B^*} N)$$

Change of rings spectral sequence:

$$Ext^*_{B^*}(M, CoTor^*_{C^*}(B^*, N)) \Longrightarrow Ext^*_{C^*}(M, N)$$

[Collapses if $B^*$ is an extended $C^*$-comodule to give $Ext^*_{B^*}(M, B^* \Box_{C^*} N) \cong Ext_{C^*}(M, N)$]

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**Note**:

$$\begin{tikzcd}
k \Box_{B^*} M \arrow[r] \arrow[d, dashed, "\cong"] & M \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\wr"] & \Gamma \otimes M \arrow[d, "\wr"] \\
Hom_{B^*}(k, M) \arrow[r] & Hom(k, M) \arrow[r, shift left] \arrow[r, shift right] & Hom(k, \Gamma \otimes M)
\end{tikzcd}$$

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commutative Hopf algebras$/k$ are group objects in affine schemes

| commutative Hopf Algebra | Group |
| --- | --- |
| $B^*$ | $G$ |
| $B^*$ comodule | $G$-module |
| $M \mathbin{\square}_{B^*} N$ | $M \otimes_{\mathbb{Z}[G]} N$ |
| $B^* \twoheadrightarrow C^*$ | $H \leq G$ |
| | $G \underset{H\text{-set}}{\cong} H \times G/H$ |
| | $\Downarrow$ |
| $B^* \cong (B^* \mathbin{\square}_{C^*} k) \otimes_k C^*$ <br> right $C^*$-comodule | $\mathbb{Z}[G] \cong \mathbb{Z}[H] \otimes_{\mathbb{Z}} (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z})$ <br> left $H$-mod |

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Change of rings iso

$B^* \twoheadrightarrow C^*$ commutative Hopf alg's

$$\mathrm{Ext}^*_{B^*}(M, B^* \mathbin{\square}_{C^*} N) \cong \mathrm{Ext}^*_{C^*}(M, N)$$

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Compare: "Shapiro's lemma"

$$H_*(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M) \cong H_*(H, M)$$

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$p=2$

$$\left(A/\!\!/A(n)\right)_*$$

$$\cong$$

$$A_* \mathbin{\square}_{A(n)_*} \mathbb{F}_2 \subset A_*$$

$$=$$

$$\left\{ x \in A_* \;\middle|\; \begin{array}{ccc} A_* \otimes A_* & \longrightarrow & A_* \otimes A(n)_* \\ \psi(x) & \longmapsto & 0 \\ - x \otimes 1 & & \end{array} \right\}$$

$$\begin{array}{ccc} & & A_* \otimes A(n)_* \\ & & \cup \\ \psi(\bar\xi_s^{2^t}) = \sum_{i_1+i_2=s} \bar\xi_{i_1}^{2^t} \otimes \bar\xi_{i_2}^{2^{i_1+t}} & \longmapsto & 0 \\ - \bar\xi_s \otimes 1 & & \end{array}$$

$\bar\xi_i = c(\xi_i)$

$$- \xi_s \otimes 1$$

$$\Updownarrow$$

$$i_1 + i_2 + t \geq n+1$$

$$\cup$$

$$s+t$$

$$\left(A/\!\!/A(n)\right)_* \cong \mathbb{F}_2\left[\bar\xi_1^{2^{n+1}}, \bar\xi_2^{2^n}, \ldots, \bar\xi_{n+1}^{2}, \bar\xi_{n+2}, \ldots\right]$$

$\cap$ sub-comodule algebra

$A_*$

Change of rings:

$$\mathrm{Ext}^*_{A_*}\left(\mathbb{F}_p, (A/\!/A(n))_*\right) \cong \mathrm{Ext}_{A(n)_*}(\mathbb{F}_p, \mathbb{F}_p)$$

More generally: given an $A(n)_*$-comodule $M$

$$\mathrm{Ext}^*_{A_*}(\mathbb{F}_p, A_* \square_{A(n)_*} M) \cong \mathrm{Ext}^*_{A(n)_*}(\mathbb{F}_p, M)$$