5 - First Computations

Note Title 9/17/2008

KU = cx K-thy spectrum

KO = real K-thy spectrum

ku = connective cover of KU

ko = connective cover of KO

$$
\begin{array}{l|l}
KU_* = \cdots 0 \ \mathbb{Z} \ 0 & \mathbb{Z} \ 0 \ \mathbb{Z} \ 0 \ - - \\
KO_* = \cdots \mathbb{Z} \ 0 \ 0 \ 0 & \mathbb{Z} \ \mathbb{Z}/2 \ \mathbb{Z}/2 \ 0 \ \mathbb{Z} \ - - -
\end{array}
$$

ku_*

ko_*

Thm: (we'll take this as input)

$$H\mathbb{F}_2^* ku = A /\!/ E[Q_0, Q_1]$$

$$H\mathbb{F}_2^* ko = A /\!/ A(1)$$

X = space

problem: compute $ku_* X = \pi_* \, ku \wedge X$

$ko_* X = \pi_* \, ko \wedge X$

Künneth $\Rightarrow$ $H_* = (H\mathbb{F}_2)_*$

$$H^*(ko \wedge X) \cong H^*ko \otimes_{\mathbb{F}_2} H^*X$$

$$\cong A/\!/A(1) \otimes H^*X \circlearrowleft \text{diagonal}$$

A-action
(Cartan formula)

Lemma: Suppose that $C \subset B$ sub Hopf algebra$/k$
M, N are B-modules

(i) $M \otimes_k N$ is a B-module using

$$\begin{tikzcd}
B \otimes_k (M \otimes_k N) \arrow[r, "\psi \otimes 1 \otimes 1"'] \arrow[d] & B \otimes_k B \otimes_k M \otimes_k N \arrow[d, "\wr"] \\
M \otimes_k N & B \otimes_k M \otimes_k B \otimes_k N \arrow[l, "\mu \otimes \mu"]
\end{tikzcd}$$

"diagonal action"

(ii)

$$\underset{\text{diagonal action}}{B/\!/C \otimes M} \overset{\text{B-modules}}{\cong} \underset{\text{left action}}{B \otimes_C M}$$

(pf of ii)

$$b = \sum_i b' \otimes b''$$

$$B/\!/C \otimes M \cong B \otimes_C M$$

$$b \otimes m \longmapsto \sum_i b' \otimes c(b'')m$$

$$\sum_i b' \otimes b'' m \longleftarrow b \otimes m$$

$\square$

<u>Consequence</u>: $ko_* X \cong A_* \square_{A(1)_*} H_* X$

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, H_*(ko \wedge X)) \Longrightarrow ko_{t-s} X^\wedge_2$$

$$\cong$$

$$\mathrm{Ext}^{s,t}_{A_*}(\mathbb{F}_2, A_* \square_{A(1)_*} H_* X)$$

$\cong$ change of rings

$$\mathrm{Ext}^{s,t}_{A(1)_*}(\mathbb{F}_2, H_* X) \cong \mathrm{Ext}_{A(1)}(H^* X, \mathbb{F}_2)$$

<u>Similarly</u>

$$\mathrm{Ext}^{s,t}_{E[\zeta_0, \zeta_1]}(\mathbb{F}_2, H_* X) \Rightarrow ku_* X^\wedge_2$$

$$\cong$$

$$\mathrm{Ext}^{s,t}_{E[Q_0, Q_1]}(H^* X, \mathbb{F}_2)$$

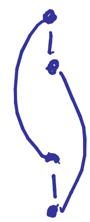
<u>e.g.</u> $X = S^0$

compute $ku_* S^0 \cong ku_*$

$$\mathrm{Ext}^{s,t}_{E[Q_0,Q_1]}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s} ku$$

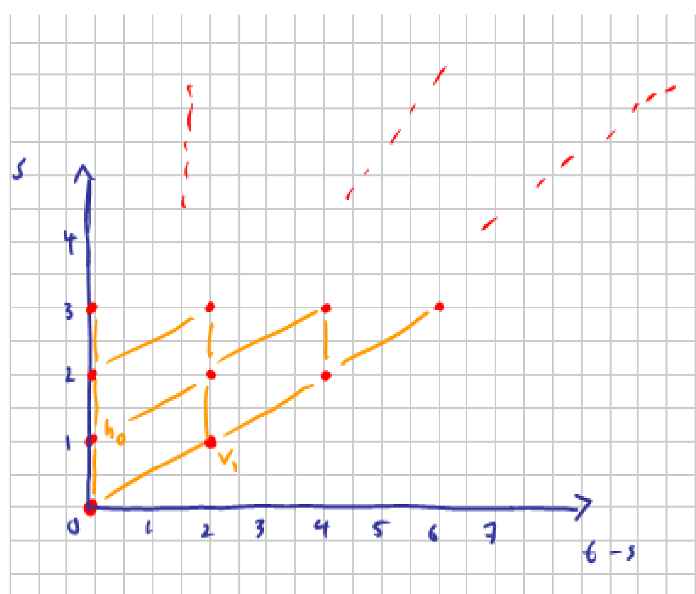
this is a spectral sequence of algebras

will return to this point ....

$E[Q_0, Q_1] =$ 

$|Q_1| = 3$

$|Q_0| = 1$



$$\mathrm{Ext}_{E[Q_0,Q_1]}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_0, v_1]$$

$$\begin{array}{cc} & \quad t-s \quad s \\ |h_0| = & (0, 1) \\ |v_1| = & (2, 1) \end{array}$$

<u>Note!</u> $h_0 \leftrightarrow Q_0$

$h_1 \leftrightarrow Q_1$

$$\left[\begin{aligned} \mathrm{Ext}^1_{B}(h,h) &= QB \\ \mathrm{Ext}^1_{B^*}(k,k) &= PB^* \end{aligned}\right]$$

__Claim:__ h_0 detects $2 \in \pi_0 ko^\wedge_2$

$$S^0 \xrightarrow{\cdot 2} S^0 \xrightarrow{u} ku$$

$$\begin{tikzcd}
2 & \tilde{2} \arrow[l, mapsto] \\
\mathbb{Z}[\beta] \arrow[u, phantom, "\in"] & (2,\beta) \arrow[l, hook'] \arrow[u, phantom, "\in"] \\
ku \arrow[u, phantom, "\pi_*"] \arrow[d] & X_1 \arrow[l] \arrow[u, phantom, "\pi_*"] \arrow[d] \\
H\mathbb{F}_2 & H\mathbb{F}_2 \wedge X_1
\end{tikzcd}$$

detects 1 $\in$ $H^* ku$ $= A /\!/ E[Q_0, Q_1]$

$\mathbb{F}_2$

__Hurewicz__

$$\begin{array}{ccc} \pi_0 X_1 & \longrightarrow & H_0 X_1 \\ \| & & \cong \\ \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/2 \end{array}$$

"generated by $\tilde{2}$"

$ku \longrightarrow H$ is H-injective

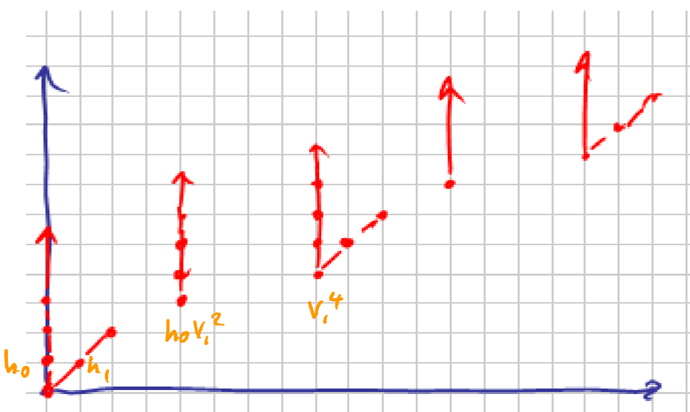
$H_* ku \hookrightarrow H_* H$

$A_* \square_{A(1)_*} \mathbb{F}_2 \hookrightarrow A_*$

No room for diff'ls $\implies$ SS converges to $\hat{\mathbb{Z}}_2[\beta]$

ko:

$$\mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}\, ko$$



h_1 detects α

$h_0 v_1^2$ detects w

v_1^4 detects v

$$\pi_* ko^\wedge_2 = \mathbb{Z}^\wedge_2[\alpha, w, v] / (2\alpha, \alpha^3, \alpha w, w^2 = 4v)$$

complex conjugation

$$KO = KU^{C_2} \hookrightarrow KU$$

$$ko \longrightarrow ku$$

$$H^* ko \longleftarrow H^* ku$$

$$A /\!/ A(1) \twoheadleftarrow A /\!/ E[Q_0, Q_1]$$

Map of ss's

$$\begin{array}{ccc} \mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) & \Longrightarrow & \pi_* ko \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{E[Q_0, Q_1]}(\mathbb{F}_2, \mathbb{F}_2) & \Longrightarrow & \pi_* ku \end{array}$$

Note: $w = \langle \alpha, 2, \alpha^2 \rangle = \langle 2, \alpha, \alpha^2 \rangle$

What?

Toda products

Massey products:

C^* is a diff'l graded algebra

$$d(xy) = d(x)y + (-1)^{|x|} x\, dy$$

$\alpha_1, \alpha_2, \alpha_3 \in H^*(C^*)$ ← graded alg

$\alpha_i = a_i$

$\alpha_1 \alpha_2 = 0$

$\alpha_2 \alpha_3 = 0$

$$\begin{array}{ccccc} a_1 & & a_2 & & a_3 \\ & b_1 & & b_2 & \end{array}$$

$$\boxed{\begin{array}{l} \bar{x} := (-1)^{1+|x|} x \\ d\bar{x}y = -\overline{dx}\,y - x\,dy \\ \bar{x}\bar{y} = -\overline{xy} \end{array}}$$

$$\left.\begin{array}{l} db_1 = \bar{a}_1 a_2 \\ db_2 = \bar{a}_2 a_3 \end{array}\right\} \implies \bar{a}_1 b_2 + \bar{b}_1 a_3 \text{ is a cocycle} \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle$$

$d(\bar{a}_1 b_2) = -a_1 \bar{a}_2 a_3$

$d(\bar{b}_1 a_3) = a_1 \bar{a}_2 a_3$

$\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a coset

Suppose b_1' b_2' satisfy $db_i' = 0$

$\Rightarrow$ $\bar{a}_1 (b_2 + b_2') + (\overline{b_1 + b_1'}) a_3$

$\shortparallel$

$\bar{a}_1 b_2' + \bar{b}_1' a_3 + (\bar{a}_1 b_2 + \bar{b}_1 a_3)$

thus

$\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is unique modulo

$\bar{\alpha}_1 H^{|\alpha_2|+|\alpha_3|-1} + \alpha_3 H^{|\alpha_1|+|\alpha_2|-1}$

$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$:

$\alpha_i \alpha_{i+1} = 0$

$\langle \alpha_i, \alpha_{i+1}, \alpha_{i+2} \rangle = 0$

$$
\begin{array}{ccccccc}
a_1 & & a_2 & & a_3 & & a_4 \\
& b_1 & & b_2 & & b_3 & \\
& & c_1 & & c_2 & &
\end{array}
$$

$$d b_i = \bar{a}_i a_{i+1}$$

$$d c_i = \bar{a}_i b_{i+1} + \bar{b}_i a_{i+2}$$

$$\bar{a}_1 c_2 + \bar{b}_1 b_3 + \bar{c}_1 a_4 \in \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$$

<u>Toda brackets</u>

$R =$ assoc. ring spectrum.

$\alpha_i \in \pi_{n_i} R$ $\qquad a_i : S^{n_i} \longrightarrow R$

$\alpha_1 \alpha_2 = 0 \qquad \alpha_2 \alpha_3 = 0$

$$\begin{tikzcd}
S^{n_i} \wedge S^{n_{i+1}} \wedge \partial I \arrow[r, "\bar{a}_i \wedge a_{i+1}"] \arrow[d, hook] & R \wedge R \arrow[dr, "\mu"] & \\
S^{n_i} \wedge S^{n_{i+1}} \wedge I \arrow[rr, "b_i"'] & & R
\end{tikzcd}$$

$$\begin{tikzcd}
S^{n_1} \wedge S^{n_2} \wedge S^{n_3} \arrow[rr] \arrow[d] & & S^{n_1} \wedge S^{n_2} \wedge S^{n_3} \wedge I \arrow[dl] \arrow[dd, "\bar{a}_1 \wedge b_2"] \\
S^{n_1} \wedge S^{n_2} \wedge I \wedge S^{n_3} \arrow[r] \arrow[drr, bend right, "\bar{b}_1 \wedge a_3"'] & S^{n_1} \wedge S^{n_2} \wedge S^{n_3} \wedge S^1 \arrow[dr, dashed] & \\
& & R
\end{tikzcd}$$

$$S^1 = I \cup_{\partial I} I$$

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e.g. $\quad F(X, X)$

$$\longrightarrow \Sigma^{|f_3|+|f_2|+|f_1|} X \xrightarrow{f_1} \Sigma^{|f_3|+|f_2|} X \xrightarrow{f_2} \Sigma^{|f_3|} X \xrightarrow{f_3} X$$

$$f_2 f_1 \simeq 0 \qquad f_3 f_2 \simeq 0$$

$$\Downarrow$$

$$\langle f_3, f_2, f_1 \rangle : S^{|f_1|+|f_2|+|f_3|+1} \longrightarrow X$$

Moss Convergence thm

$$\mathrm{Ext}_{E_*E}(E_*, E_*) \Longrightarrow \pi_* S^1_E$$

massey products in the $\Longrightarrow$ Toda brackets here

$C^*_{E_*E}(E_*)$ is a Diff'l graded alg.

$$[\gamma_1 | \cdots | \gamma_{s_1}] \cdot [\delta_1 | \cdots | \delta_{s_2}]$$

$$= \pm [\gamma_1 | \cdots | \gamma_{s_1} | \delta_1 | \cdots | \delta_{s_2}]$$
