6 - May Spectral sequence

Note Title 9/24/2008

Recall $I \longrightarrow A \longrightarrow k$

$\{I^s\}$ = decreasing filtration of A

E^0A = associated graded (primitively generated)

$$\cong V(PE^0A)$$

Milnor-Moore

$$PE^0A = \begin{cases} \mathbb{F}_2\{P^t_s\} & p=2 \\ \mathbb{F}_p\{P^t_s\} \oplus \Lambda_{\mathbb{F}_p}[Q_i] & p>2 \end{cases}$$

Dual filtration

$$I^{\otimes s} \xrightarrow{\mu_s} A \longrightarrow A/I^s \longrightarrow 0$$

$$0 \longrightarrow (A/I^s)_* \longrightarrow A_* \xrightarrow[\Psi_s]{} I_*^{\otimes s}$$

$$(A/I^s)_* \longrightarrow A_* \longrightarrow (I^s)_*$$

get increasing filtration on A_*

$$x \in \overset{\ker \psi_{s+1}}{\overset{\shortparallel}{F_s}} A_* \iff \psi_s(x) = 0$$

e.g. $\psi(\xi_1^{p^t}) = \xi_1^{p^t} \otimes 1 + 1 \otimes \xi_1^{p^t} = 0$

in $I_2^{\otimes 2}$

$$\implies |\xi_1^{2^t}|_{may} = 1$$

$$\psi(\xi_2^{p^t}) = \xi_2^{p^t} \cancel{\otimes} 1 + \xi_1^{p^{t+1}} \otimes \xi_1^{r} + 1 \cancel{\otimes} \xi_2^{p^{t+1}}$$

$$\psi_2(\xi_2^{p^t}) = 0$$

$$\implies |\xi_2^{p^t}|_{may} = 2$$

$$|\xi_s^{p^t}|_{may} = s \qquad |\check{\tau}_i|_{may} = i+1$$

Get filtration on $C^*_{A_*}(\mathbb{F}_p)$

Get a spectral sequence

$$E_2^{s,t,v} = \mathrm{Ext}^{s,t,v}_{E^0A}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow \mathrm{Ext}^{s,t}_{E^0A}(\mathbb{F}_p, \mathbb{F}_p)$$

$$d_r : E_r^{s,t,v} \longrightarrow E_r^{s+1,t,v-r+1}$$

$$E^0A = V(\overset{\text{restriction}=0}{P}E^0A) \underset{\substack{\uparrow \\ \text{additively}}}{=} \begin{cases} \Lambda_{\mathbb{F}_2}[P_s^t] & p=2 \\ \mathrm{Tr}_{\mathbb{F}_p}[P_s^t] \otimes \Lambda_{\mathbb{F}_p}[Q_i] & p>2 \end{cases}$$

"Koszul spectral sequence"

Increasing filtration on E^0A by monomial length

$$E^0E^0A \underset{\substack{\uparrow \\ \text{multiplicatively}}}{=} \{ \text{ same } \}$$

KSS

$$\overset{\overset{E_2^{KSS}}{\shortparallel}}{\mathrm{Ext}^{s,t}_{E^0E^0A}(\mathbb{F}_p, \mathbb{F}_p)} \Longrightarrow \overset{\overset{E_2^{May}}{\shortparallel}}{\mathrm{Ext}^{s,t}_{E^0A}(\mathbb{F}_p, \mathbb{F}_p)}$$

Fortunately, we have **Prop** (May): the KSS collapses at E_3

E_2^{KSS} : Fundamental Computations

Recall $\mathrm{Ext}^{**}_{\Lambda_{\mathbb{F}_p}[Q_0,Q_1]}(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p[v_0, v_1]$

$|v_0| = (1, |Q_0|)$

$|v_1| = (1, |Q_1|)$

In general

$$\mathrm{Ext}^{**}_{\Lambda_{\mathbb{F}_p}[x]}(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p[y]$$

$|x| =$ odd if p odd

$|y| = (1, |x|)$

p odd

$$\mathrm{Ext}^{**}_{\mathrm{Tr}_{\mathbb{F}_p}[x]}(\mathbb{F}_p,\mathbb{F}_p) = \mathbb{F}_p[a] \otimes \Lambda_{\mathbb{F}_p}[b]$$

$|x| =$ even

$|a| = (1, |x|)$

$|b| = (2, p|x|)$

Note

$\mathrm{Tr}_{\mathbb{F}_p}[x] \cong \mathbb{F}_p[C_p]$

$\langle \sigma \rangle = C_p$ $\quad (\sigma - 1)^p = 0$

$\sigma - 1 = x$

Ext of modules or comodules?
actually both

Lem If we let $\Lambda_{\mathbb{F}_p}[x]$, $\mathrm{Tr}_{\mathbb{F}_p}[x]$ be Primitively generated: $\psi(x) = x \otimes 1 + 1 \otimes x$

Thm:

$$\left(\Lambda_{\mathbb{F}_p}[x]\right)^* \cong \Lambda_{\mathbb{F}_p}[x]$$

$$\left(Tr_{\mathbb{F}_p}[x]\right)^* \cong Tr_{\mathbb{F}_p}[x]$$

So we can also regard them as computing of Ext of comodules.

With this latter point of view:

In

$\Lambda_{\mathbb{F}_p}[x]$:

$$y = [x] \in C^1_{\Lambda_{\mathbb{F}_p}[x]}(\mathbb{F}_p)$$

$Tr_{\mathbb{F}_p}[x]$:

$$a = [x] \in C^1_{Tr_{\mathbb{F}_p}[x]}$$

$$b = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i | x^{p-i}$$

$$d[x^2] = -2[x|x]$$

Note: $b = \langle \underbrace{a, a, \dots, a}_{p} \rangle$

e.g. $p = 3$

$$[x] \quad [x] \quad [x]$$

$$[-\tfrac{1}{2}x^2] \quad [-\tfrac{1}{2}x^2]$$

$$-\tfrac{1}{2}[x|x^2] - \tfrac{1}{2}[x^2|x]$$

$p = 5$

$$x \qquad x \qquad x \qquad x \qquad x$$

$$-\frac{1}{2}x^2 \qquad -\frac{1}{2}x^2 \qquad -\frac{1}{2}x^2 \qquad -\frac{1}{2}x^2$$

$$\frac{1}{2\cdot 3}x^3 \qquad \frac{1}{2\cdot 3}x^3 \qquad \frac{1}{2\cdot 3}x^3$$

$$-\frac{3}{2}x^4 \qquad -\frac{3}{2}x^4$$

Thm $p=2$

$$E_2^{KSS} = \mathrm{Ext}^{**}_{\Lambda[\xi_i^{2^j}]}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\left[h_{ij} \,\middle|\, \begin{matrix} i \geq 1 \\ j \geq 0 \end{matrix}\right]$$

$$h_{ij} = [\xi_i^{p^j}] \in C^1_{E^0E^0A_*}$$

$$|h_{ij}| = (1,\ 2^j(2^i - 1))$$

p odd

$$E_2^{KSS} = \mathrm{Ext}^{**}_{Tr[\xi_i^{p^j}] \otimes \Lambda[\tau_i]} = \Lambda\left[h_{ij} \,\middle|\, \begin{matrix} i \geq 1 \\ j \geq 0 \end{matrix}\right]$$

$$\otimes\ \mathbb{F}_p\left[b_{ij} \,\middle|\, \begin{matrix} i \geq 1 \\ j \geq 0 \end{matrix}\right]$$

$$\otimes\ \mathbb{F}_p\left[v_n \,\middle|\, n \geq 0\right]$$

$$|h_{ij}| = (1,\ 2p^j(p^i - 1))$$

$$|b_{ij}| = (2,\ 2p^{j+1}(p^i - 1))$$

$$|v_n| = (1,\ 2p^n - 1)$$

Computing d_2^{KSS}

in $C^*_{E_*A}$:

$p=2$

$$d[\xi_i^{2^j}] = \sum_{i_1+i_2=i}' \xi_{i_1}^{2^{j+i_2}} | \xi_{i_2}^{2^j}$$

$$\Longrightarrow \quad d h_{ij} = \sum_{i_1+i_2}' h_{i_1, j+i_2} h_{i_2,j}$$

$p = \text{odd}$

$$d[\xi_i^{p^j}] = -\sum_{i_1+i_2=i} \xi_{i_1}^{p^{j+i_2}} | \xi_{i_2}^{p^j}$$

$$\Longrightarrow \quad d_2 h_{ij} = -\sum_{i_1+i_2=i}' h_{i_1,i_2+j} h_{i_2,j}$$

$d_2 b_{ij} = 0$ "little tricky"

(analogous to fact at $p=2$ that $d_2 h_{ij}^2 = 0$)

$h_{ij}^2 \longleftrightarrow b_{ij}$

$$d(\tilde{\tau}_n) = -\sum_{n_1+n_2=n}' \left[\xi_{n_1}^{p^{n_2}} \middle| \tilde{\tau}_{n_2}\right]$$

$$\Longrightarrow \quad d_2 v_n = -\sum_{n_1+n_2=n}' h_{n_1,n_2} v_{n_2}$$

Thus, we may regard (E_2^{kss}, d_2) ← refer to this as E_1^{MSS} as an explicit diff'l graded alg

$$H^*(E_2^{kss}, d_2) \cong E_2^{MSS}$$

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e.g. $p=2$

$$A(1)_* = \mathbb{F}_2[\xi_1, \xi_2]/(\xi_1^4, \xi_2^2)$$

$$E_2^{kss} \Longrightarrow E_2^{MSS} \Longrightarrow \mathrm{Ext}_{A(1)_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_* ko_2^\wedge$$

$$E_2^{kss} = E_1^{MSS} = \mathbb{F}_2[h_{1,0}, h_{1,1}, h_{2,0}]$$

$$d_1 h_{1,0} = d_1 h_{1,1} = 0$$

$$d_1 h_{2,0} = h_{1,1} h_{1,0}$$

$$d_1 h_{2,0}^2 = 0$$

$$b_{2,0} := h_{2,0}^2$$

$$h_0 = h_{1,0}$$

$$h_1 = h_{1,1}$$

general notation:

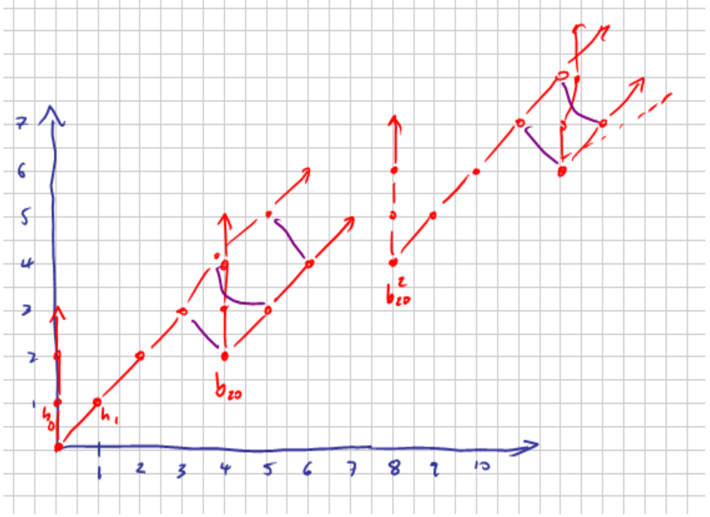
$h_{1,i} =: h_i$

always a cycle because $\xi_1^{2^i}$ primitive

E_2^{MSS}

May d_2:

$$d_2 b_{20} = h_1^3$$



$$h_{20} \longleftrightarrow [\xi_2]$$

$$b_{20} \longleftrightarrow h_{20}^2 \longleftrightarrow [\xi_2|\xi_2]$$

$$d[\xi_2|\xi_2] = \underbrace{[\xi_1^2|\xi_1|\xi_2]}_{4} + \underbrace{[\xi_2|\xi_1^2|\xi_1]}_{4}$$

May filt 4

Koszul filt 2

$$\underset{\substack{4\\3}}{d[\xi_1^2|\xi_1\xi_2]} = [\underbrace{\xi_1^2|\xi_1}_{4}|\xi_2] + [\underbrace{\xi_1^2|\xi_2}_{4}|\xi_1] + [\underbrace{\xi_1^2|\xi_1^3}_{4}|\xi_1] + \underset{3}{\boxed{[\xi_1^2|\xi_1^2|\xi_1^2]}}$$

$$\underset{\substack{4\\3}}{d[\xi_2\xi_1^2|\xi_1]} = [\underbrace{\xi_2|\xi_1^2}_{4}|\xi_1] + [\underbrace{\xi_1^2|\xi_2}_{4}|\xi_1] + [\underbrace{\xi_1^2|\xi_1^3}_{4}|\xi_1] + \cancelto{0}{[\xi_1^4|\xi_1|\xi]}$$

$$\boxed{\xi_1^4 = 0 \text{ in } A(1)_*}$$

<u>So this is saying</u>

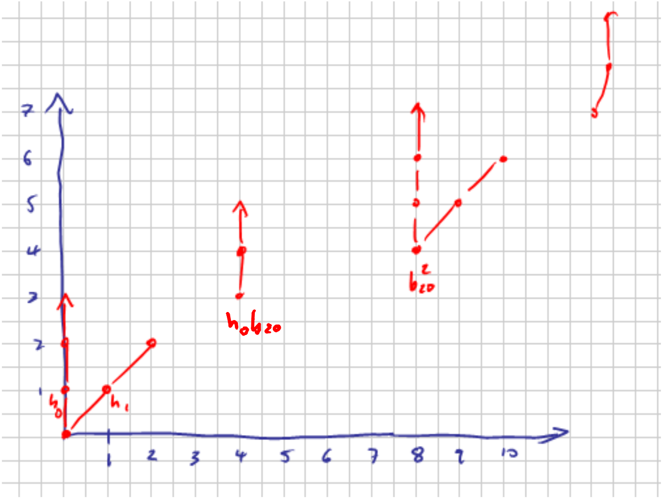
$$b_{20} = [\xi_2|\xi_2] + [\xi_1^2|\xi_1\xi_2] + [\xi_2\xi_1^2|\xi_1] \quad \in E_2^{MSS}$$

<u>and</u> $= h_{20}^2$ mod <u>higher</u> Koszul filtration

$$\underset{\text{May filt } 4}{d}(b_{20}) = 0 \quad \text{mod May filt } 3$$

$$d_2(b_{20}) = \underset{\text{in May filt } 3}{h_1^3}$$

$$E_3^{MSS} = E_\infty^{MSS} = E_\infty^{ASS}$$



$$\pi_* ko^\wedge_2 = \mathbb{Z}^\wedge_2[\eta, w, v] \big/ (2\eta, \eta^3, w^2 = 4v, \eta w)$$

h_1	$\lvert\eta\rvert = 1$
$h_0 b_{20}$	$\lvert w\rvert = 4$
b_{20}^2	$\lvert v\rvert = 8$

$(h_0 = 2)$

Rmk: In E_2^{MSS}

$$\begin{array}{ccc} h_0 & h_1 & h_1^2 \\ & 0 & b_{20} \end{array}$$

$\rightsquigarrow$ $h_0 b_{20}$ detects elt of $\langle h_0, h_1, h_1^2 \rangle$

$= \langle 2, \eta, \eta^2 \rangle$

Why can I do this??

In general, this is allowed in the absence of a "crossing diff'l"

Crossing diff'ls