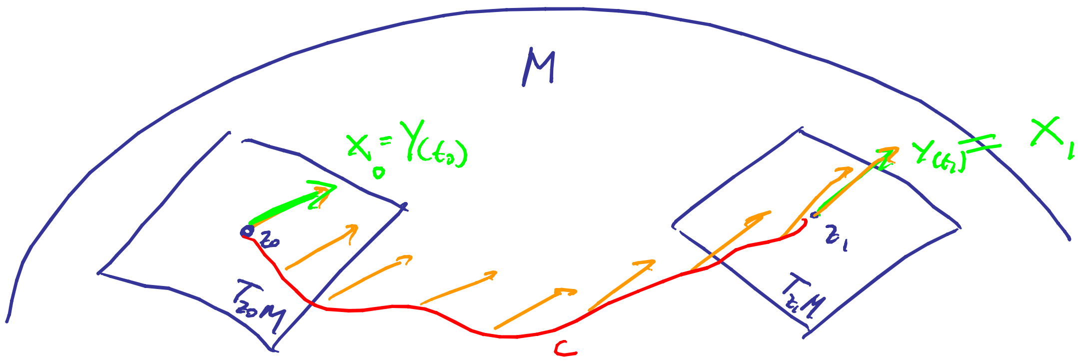
15 Parallel displacement + geodesics

Note Title 10/29/2009

$$M \hookrightarrow \mathbb{R}^{n+1}, \quad \nabla$$



Given a curve $c : [t_0, t_1] \longrightarrow M$

$$c(t_i) = z_i$$

get map

$$P_c : T_{z_0}M \longrightarrow T_{z_1}M$$

$$X_0 \longmapsto X_1$$

Q: does this map depend on curve?

Q: does this map exist, is it well defined?

Thm Given $X_0 \in T_{z_0}M$

$c : [t_0, t_1] \longrightarrow M$ curve

$c(t_i) = z_i$

$\exists!$ parallel vector field $Y(t)$ along c

s.t. $Y(t_0) = X_0$

(Define $P_c X_0 = Y(t_1)$)

(Pf) Write c in local coordinates

$c = (u_1(t), \ldots, u_n(t))$

$\dot{c} = \sum \frac{du_i}{dt} \frac{\partial f}{\partial u_i}$

$$0 = \nabla_{\dot{c}} Y = \sum_{i,k} \frac{du_i}{dt}\left(\frac{\partial Y^k}{\partial u_i} + \sum_j Y^j(t)\, \Gamma^k_{ij}(c(t))\right)\frac{\partial f}{\partial u^k}$$

$$= \sum_k \left(\frac{dY^k}{dt} + \sum_{i,j} \frac{du_i}{dt} Y^j(t)\, \Gamma^k_{ij}\right)\frac{\partial f}{\partial u_k}$$

i.e.

$$\frac{d}{\partial t}\begin{bmatrix} Y^1 \\ \vdots \\ Y^n \end{bmatrix} = \left[\sum_i \frac{\partial u_i}{\partial t}\Gamma^k_{ij}\right]_{k,j}\begin{bmatrix} Y^1 \\ \vdots \\ Y^n \end{bmatrix}$$

$$\frac{\partial}{\partial t} Y(t) = A(t)\, Y(t)$$

Linear O.D.E

$\Longrightarrow$ has a solution.