# 16 - Thm Egregium

Note Title 11/3/2009

(warning: pset 7 may include 4C)

**Q:** given $h_{ij}$, $g_{ij}$

is $f$ determined?

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$$\begin{cases} D_X Y = \nabla_X Y + \mathrm{II}(X,Y)\,\nu \\ D\nu = -L \circ Df \end{cases}$$

$$D_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} = \nabla_{\frac{\partial f}{\partial u_i}} \frac{\partial f}{\partial u_j} + \mathrm{II}\left(\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right)\nu$$

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial u_k} + h_{ij}\,\nu$$

$$h_i^k = \sum_l g^{kl} h_{l,i}$$

$$\frac{\partial \nu}{\partial u_i} = -L \frac{\partial f}{\partial u_i}$$

$$= -\sum h_i^k \frac{\partial f}{\partial u_k}$$

$\nu$ depends on $f$

**Q:** Can we use the above system of partial differential equations to "solve for $f$"

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## Existence of sol'ns to single P.D.E

given functions

$$g_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\vdots$$

$$g_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

when is there an $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

s.t. $\dfrac{\partial f}{\partial u_i} = g_i \quad \forall i$

$$\frac{\partial}{\partial u_j}\frac{\partial f}{\partial u_i} = \frac{\partial g_i}{\partial u_j}$$

$$\parallel$$

$$\frac{\partial}{\partial u_i}\frac{\partial f}{\partial u_j} = \frac{\partial g_j}{\partial u_i}$$

"integrability conditions"

Turns out this is enough!

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**similarly** $$\frac{\partial^3 f}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3 f}{\partial u_i \partial u^k \partial u^j}$$

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$$0 = \frac{\partial}{\partial u_k} \frac{\partial^2 f}{\partial u_i \partial u_j} - \frac{\partial}{\partial u_j} \frac{\partial^2 f}{\partial u_i \partial u_k}$$

$$= \sum_l \left[ \frac{\partial}{\partial u_k}\left( \Gamma_{ij}^l \frac{\partial f}{\partial u_l} \right) - \frac{\partial}{\partial u_j}\left( \Gamma_{ik}^l \frac{\partial f}{\partial u_k} \right) \right]$$

$$+ \left( \frac{\partial}{\partial u_k}\left( h_{ij} \nu \right) - \frac{\partial}{\partial u_j}\left( h_{ik} \nu \right) \right)$$

$$= \sum_l \left[ \frac{\partial}{\partial u_k} \Gamma_{ij}^l \frac{\partial f}{\partial u_l} - \frac{\partial}{\partial u_j} \Gamma_{ik}^l \frac{\partial f}{\partial u_l} + \Gamma_{ij}^l \frac{\partial^2 f}{\partial u_k \partial u_l} \right.$$

$$\left. - \Gamma_{ik}^l \frac{\partial^2 f}{\partial u_j \partial u_l} \right] + \left( \frac{\partial}{\partial u_k} h_{ij} - \frac{\partial}{\partial u_j} h_{ik} \right) \nu$$

$$+ h_{ij} \frac{\partial \nu}{\partial u_k} - h_{ik} \frac{\partial \nu}{\partial u_j}$$

$$= \sum_l \left[ \left( \frac{\partial}{\partial u_k} \Gamma_{ij}^l - \frac{\partial}{\partial u_j} \Gamma_{ik}^l \right) \frac{\partial f}{\partial u_l} + \sum_m \left( \Gamma_{ij}^l \Gamma_{k,l}^m - \Gamma_{ik}^l \Gamma_{jl}^m \right) \frac{\partial f}{\partial u_m} \right.$$

$$\left. + \left( \Gamma_{ij}^l h_{kl} - \Gamma_{ik}^l h_{jl} \right) \nu \right]$$

$$+ \left( \frac{\partial}{\partial u_k} h_{ij} - \frac{\partial}{u_j} h_{ik} \right) \nu$$

$$\sum_l \left( -h_{ij} h_k^l + h_{ik} h_j^l \right) \frac{\partial f}{\partial u_l}$$

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interchange $m, l$

So: $\frac{\partial f}{\partial u_l}$ term Gauss Eqn

$$\boxed{\frac{\partial}{\partial u_k} \Gamma_{ij}^l - \frac{\partial}{\partial u_j} \Gamma_{ik}^l + \sum_m \Gamma_{ij}^m \Gamma_{k,m}^l - \Gamma_{ik}^m \Gamma_{jm}^l = h_{ij} h_k^l - h_{ik} h_j^l}$$

$\nu$ term Codazzi-Mainardi Eqn

$$\boxed{\frac{\partial}{\partial u_k} h_{ij} - \frac{\partial}{\partial u_j} h_{ik} + \sum_l \left( \Gamma_{ij}^l h_{kl} - \Gamma_{ik}^l h_{jl} \right) = 0}$$

Curvature tensor:

$$R_{ikj}^{\ \ \ s} = \frac{\partial}{\partial u_k}\Gamma_{ij}^{s} - \frac{\partial}{\partial u_j}\Gamma_{ik}^{s} + \sum_r \left(\Gamma_{ij}^{r}\Gamma_{rk}^{s} - \Gamma_{ik}^{r}\Gamma_{r,j}^{s}\right)$$

M surface

$i = j = 1 \qquad k = 2$

$$\frac{\partial}{\partial u_2}\Gamma_{11}^{l} - \frac{\partial}{\partial u_1}\Gamma_{12}^{l} + \sum_m \Gamma_{11}^{m}\Gamma_{2,m}^{l} - \Gamma_{12}^{m}\Gamma_{1,m}^{l}$$

$$= h_{11} h_2^{l} - h_{12} h_1^{l}$$

$h_{i}^{t}$ w/ $g_{l,t}$

$$\frac{\partial}{\partial u_2}\Gamma_{11}^{l} - \frac{\partial}{\partial u_1}\Gamma_{12}^{l} + \sum_m \Gamma_{11}^{m}\Gamma_{2m,t} - \Gamma_{12}^{m}\Gamma_{1m,t}$$

$$= h_{11} h_{2,t} - h_{12} h_{1,t} \overset{t=2}{=} \operatorname{Det} II \ !$$

## Thm Egregium

$K$ only depends on $I(-,-)$

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Coordinate free approach

$$D_X Y = \nabla_X Y + \langle LX, Y\rangle N$$

$$L = -DN$$

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$$\frac{\partial^3 f}{\partial u_i \partial u_j \partial u_k} = \frac{\partial^3}{\partial u_j \partial u_i \partial u_k} f \qquad X = \frac{\partial f}{\partial u_i} \quad Y = \frac{\partial f}{\partial u_j} \quad Z = \frac{\partial f}{\partial u_k}$$

$$D_X D_Y Z = D_Y D_X Z$$

Lemma $X, Y, Z$ vector fields

$$D_X D_Y Z - D_Y D_X Z = D_{[X,Y]} Z$$

Pf

$$\varphi$$

$$D_X D_Y \varphi$$

write: $X = \sum_i X^i \frac{\partial f}{\partial u_i}$

$Y = \sum_i Y^i \frac{\partial f}{\partial u_i}$

$$D_X D_Y \varphi = D_X \sum_i Y^i \frac{\partial \varphi}{\partial u_i}$$

$$= \sum_j X^j \left( \frac{\partial Y^i}{\partial u_j} \frac{\partial \varphi}{\partial u_i} + Y^i \frac{\partial^2 \varphi}{\partial u_i \partial u_j} \right)$$

$$D_X D_Y \varphi - D_Y D_X \varphi = \sum_j \left( X^j \frac{\partial Y^i}{\partial u_j} - Y^j \frac{\partial X^i}{\partial u_j} \right) \frac{\partial \varphi}{\partial u_i}$$

$$= D_{[X,Y]} \varphi$$

$\square$

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Gauss Eqn

$$0 = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

$$= D_X \nabla_Y Z - D_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$+ D_X \left( \langle LY, Z \rangle N \right) - D_Y \left( \langle LX, Z \rangle N \right)$$

$$- \langle L[X,Y], Z \rangle N$$

$$\left( \begin{aligned} &L = -DN \\ &\Rightarrow LX = -D_X N \end{aligned} \right)$$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$+ \cancel{\langle L \nabla_Y Z, X \rangle N} - \cancel{\langle L \nabla_X Z, Y \rangle N}$$

$$+ \underline{\langle \nabla_X LY, Z \rangle} N + \cancel{\langle LY, \nabla_X Z \rangle N}$$

$$- \langle LY, Z \rangle LX$$

$$- \underline{\langle \nabla_Y LX, Z \rangle} N - \cancel{\langle LX, \nabla_Y Z \rangle N}$$

$$+ \langle LX, Z \rangle LY - \underline{\langle L[X,Y], Z \rangle} N$$

So:

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z =$$

$$\langle LY, Z\rangle LX - \langle LX, Z\rangle LY$$

$$= \mathrm{II}(Y,Z)\, LX - \mathrm{II}(X,Z)\, LY$$

and

$$\nabla_X LY - \nabla_Y LX = L[X,Y]$$

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Curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$R(X,Y)Z = \mathrm{II}(Y,Z)\, LX - \mathrm{II}(X,Z)\, LY$$

Consequence $R(X,Y)Z(z_0)$ only depends on

$$\big(X(z_0),\ Y(z_0),\ Z(z_0)\big)$$

## Aside What is a tensor?

$$A_{z_0} : \overbrace{T_{z_0}M \times \cdots \times T_{z_0}M}^{n} \longrightarrow T_{z_0}M \qquad \text{deg } (1,s)$$

$$n \qquad\qquad \mathbb{R} \qquad \text{deg } (0,s)$$

$A(X_1, \dots, X_n)$ bilinear in each variable

e.g.

$$\left.\begin{array}{l} \mathrm{I}(-,-) \\ \mathrm{II}(-,-) \end{array}\right\} \text{deg } (0,2) \qquad \begin{array}{l} g_{ij} \\ h_{ij} \end{array}$$

whereas

$$R(-,-)- \quad \} \text{ deg } (1,3) \qquad R_{ijk}^{\ell}$$

$$\langle R(-,-)-, - \rangle . \qquad \text{deg } (0,4) \qquad R_{ijkl}$$

Note: linear

$$A_{z_0} : \underbrace{T_{z_0}M \otimes T_{z_0}M \otimes \cdots \otimes T_{z_0}M}_{s} \longrightarrow T_{z_0}M$$

given $A$ (type $(1,s)$) $A^{j}_{i_1 \dots i_s}$

$B$ (type $(0,1)$) $B_j$

get tensor of type $(0,s)$

$$B(A(x_1, \dots, x_s)) = C(x_1, \dots, x_s)$$

obtained by $C_{i_1 \dots i_s} \sim \sum_j A^{j}_{i_1 \dots i_s} B_j$

"contraction"

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e.g. $R(X,Y)Z \sim R^{l}_{ijk}$

$$I(R(X,Y)Z, W) \sim R_{ijk}^{l}$$

$$I(-,-) \sim g_{ij}$$

$$R_{ijk,l} = \sum_m g_{l,m} R^{m}_{ijk}$$

Thm Egregium (take II) $M \subseteq \mathbb{R}^3$ surface

Suppose $(X, Y)$ is an orthonormal basis of $T_{x_0}Y$

$$\langle R(X,Y)Y, X \rangle = \operatorname{Det}(L) = K$$

$$LX \sim \begin{pmatrix} \langle LX, X \rangle & \langle LY, X \rangle \\ \langle LX, Y \rangle & \langle LY, Y \rangle \end{pmatrix}$$

$$\langle R(X,Y)Y, X \rangle = \langle \langle LY, Y \rangle LX - \langle LX, Y \rangle LY, X \rangle$$

$$= \langle LY, Y \rangle \langle LX, X \rangle - \langle LX, Y \rangle \langle LY, X \rangle$$

$$= \det(L)$$

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Remark $\qquad M$ hypersurface

$X_1, \ldots, X_n$ principle curvature directions (orthonormal basis)

$LX_i = \kappa_i$

$$\langle R(X_i, X_j) X_j, X_i \rangle$$

$$= \langle \langle L X_j, X_j \rangle L X_i - \langle L X_i, X_j \rangle L X_j, X_i \rangle$$

$$= \kappa_j \kappa_i \qquad i \neq j$$

"sectional curvature of $X_i X_j$ plane"

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But! $X_1, \ldots, X_n$ is <u>any</u> orthonormal basis of $T_p M$

$$\boxed{\frac{1}{\binom{n}{2}} \sum_{i \neq j} \kappa_i \kappa_j = \frac{1}{\binom{n}{2}} \sum \langle R(X_i, X_j) X_j, X_i \rangle}$$

$$\|$$

$K_2$ "Scalar curvature"

only depends on $I(-,-)$

Note $n=1$ does not give us anything

## Consequence of Thm egregium, etc

As long as we have $g_{ij}$

get $\Gamma_{ij}^k \Longrightarrow$ have $\nabla_X Y$

and have $K$ $(n=2)$

$(K_2 \quad n>2)$

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Note: $g_{ij}$ does not define surface (e.g. cylinder)
or $h_{ij}$

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Next idea: given $h_{ij}$, we can get back the surface!

**Thm** $U \subset \mathbb{R}^2$ open

given:

- $g_{ij} : U \to \mathbb{R}$ $\quad [g_{ij}]$ symm, + def
- $h_{ij} : U \to \mathbb{R}$ $\quad [h_{ij}]$ symm
- $\bar{u} \in U$
- $\bar{p} \in \mathbb{R}^{n+1}$
- $\bar{X}_1, \dots, \bar{X}_n \in \mathbb{R}^{n+1}$
  s.t. $\langle \bar{X}_i, \bar{X}_j \rangle = g_{ij}(\bar{u})$
- $\bar{\nu} \perp \{X_i\}$

define $\Gamma_{ij}^k$ from $g_{ij}$

(*) assume Gauss and Codazzi-Mainardi equs are satisfied.

$\exists$ $V \subset U$
$\quad \ \, \bar{u} \in V$

and $f : V \longrightarrow \mathbb{R}^{n+1}$

s.t.

(1) $\overset{M}{f(V)} \subset \mathbb{R}^{n+1}$ is a hypersurface

(2) $f(\bar{u}) = \bar{p}$

(3) $\frac{\partial f}{\partial u_i} = \bar{X}_i$

(4) $\nu(\bar{u}) = \bar{\nu}$

(5) $g_{ij}$, $h_{ij}$ are I, II of M.

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(pf) existence, uniqueness of sol'n to linear PDE given "integrability conditions"

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