18 - Differential forms

Note Title 11/17/2009

Differential forms

Def Let M be a n-submfld of $\mathbb{R}^N$

A differential k-form is a tensor $(0,k)$

$$\omega_z : T_zM^{\times k} \longrightarrow T_zM$$

- smooth in z
- alternating:

$$\omega_z(X_1, \ldots X_i, \ldots, X_j, \ldots X_k) = -\,\omega_z(X_1, \ldots, X_j, \ldots, X_i \ldots X_k) \quad i \neq j$$

e.g. du^i are 1-forms

e.g. $M \subseteq \mathbb{R}^3$ is an orientable surface

$N : M \to \mathbb{R}^3$ unit normal vector field

$$\omega(X,Y) = \langle X \times Y, N \rangle$$

ω is called "dA" area form

$\otimes$ product

Local coordinates

α tensor of type $(0,k)$ $\quad \alpha \otimes \beta$

β tensor of type $(0,l)$

Wedge product $\quad \alpha = k$-form

$\beta = l$-form

$\alpha \wedge \beta = k+l$ form

$$\alpha \wedge \beta (X_1, \dots, X_{k+l}) = \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \alpha(X_{\sigma(1)} \cdots X_{\sigma(k)})\, \beta(X_{\sigma(k+1)} \cdots X_{\sigma(l)})$$

e.g. $du^{i_1} \wedge \dots \wedge du^{i_k}$ is a k-form

$$du^{i_1} \wedge \dots \wedge du^{i_p} \wedge \dots \wedge du^{i_q} \wedge \dots \wedge du^{i_k}$$

$$= (-1)\, du^{i_1} \wedge \dots \wedge du^{i_q} \wedge \dots \wedge du^{i_p} \wedge \dots \wedge du^{i_k}$$

$\Rightarrow$ if any two indices repeat

$\omega = -\omega \Rightarrow \omega = 0$

1-forms on $\mathbb{R}^3$: $A\,dx + B\,dy + C\,dz$

2-forms on $\mathbb{R}^3$: $A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$

3-forms on $\mathbb{R}^3$: $A\,dx\wedge dy\wedge dz$

there are no non-zero 4-forms on $\mathbb{R}^3$!

0-forms = functions

More generally any k-form on M an n-mfld is zero if $k > n$

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e.g.

$$\left\langle \frac{\partial f}{\partial u_i} \times \frac{\partial f}{\partial u_j}, N \right\rangle = 0 \quad \text{unless } i \neq j$$

$$\left\langle \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}, N \right\rangle = \frac{\left\langle \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2}, \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\rangle}{\left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\|}$$

other way around

$$-\sqrt{\det g_{ij}} \qquad = \left\| \frac{\partial f}{\partial u_1} \times \frac{\partial f}{\partial u_2} \right\| = \sqrt{\det g_{ij}}$$

as a tensor

get $dA = \sqrt{\det[g_{ij}]}\, du_1 \otimes du_2 - \sqrt{\det[g_{ij}]}\, du_2 \otimes du_1$

$= \sqrt{\det[g_{ij}]}\, du_1 \wedge du_2$

as a form "area form"

More generally $N \hookrightarrow M$

submfld

We can talk about "k-forms on M along N"

(compare "vector field along a curve")

Idea! $\omega = \sum \omega_{i_1,\dots,i_k}(y)\, du_{i_1} \wedge \dots \wedge du_{i_k}$

$y \in N$

Integration

Given

$$N \underset{k\text{-submfld}}{\hookrightarrow} M$$

ω a k-form on M

(or more generally, a k-form on M along N)

Can form $\displaystyle\int_N \omega \in \mathbb{R}$ (if integral converges)

Idea parametrize M w/ $(u_1, \dots, u_n)$

parametrize N w/ $(t_1, \dots, t_k)$

i.e. $\quad u_i = u_i(t_1, \dots, t_k)$

$$\omega = \sum' \omega_{i_1 \dots i_k} \, du_{i_1} \wedge \dots \wedge du_{i_k}$$

$$\omega = \text{functn of } t_1 \dots t_k$$

$$du_{i_j} = \sum_\ell \frac{\partial u_{i_j}}{\partial t_\ell} \, dt_\ell$$

get

$$H(t_1, \dots, t_k) \, dt_1 \wedge \dots \wedge dt_k$$

some function of $t_1, \dots, t_k$

$$\int \omega := \int H(t_1, \dots, t_k) \, dt_1 \dots dt_k$$

integral over an open subset of $\mathbb{R}^k$

e.g. Line integrals

$\omega = \sum_i \omega_i \, du^i$ 1-form

$c : [a,b] \longrightarrow M$ curve

write $u^i = u^i(t)$

$\omega_i = \omega_i(t)$

$$\int_c \omega = \int_c \sum_i \omega_i \, du^i$$

$$= \int_a^b \underbrace{\sum_i \omega_i(t) \frac{du^i}{dt}}_{\text{just a function.}} dt$$

Note: If X is a vect field in $\mathbb{R}^3$

$c : I \longrightarrow \mathbb{R}^3$

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$$

write $\omega = P\,dx + Q\,dy + R\,dz$

$$\int_c \omega = \int_c X \cdot d\vec{s}$$ "18.02 line integral"

Differentiation $\qquad M \hookrightarrow \mathbb{R}^{n+1}$

hypersurface

ω = k-form on M

(or more generally a k-form on $\mathbb{R}^{n+1}$ along M)

We can define: $\qquad$ (X a vector field on M)

Directional Derivative

$$D_X \omega = k\text{-form on } \mathbb{R}^{n+1} \text{ along } M$$

Covariant Derivative

$$\nabla_X \omega = k\text{-form on } M$$

Exterior Derivative

$$d\omega = (k+1)\text{-form on } M$$

Only interested in exterior derivative for now

$k=0$, $k=1$

$\underline{k=0}$ 0-form = function

$\underline{\underline{\text{abstract}}}$

$$\varphi : M \longrightarrow \mathbb{R} \qquad \text{function}$$

$$d\varphi_{z_0} : T_{z_0}M \longrightarrow \mathbb{R} \qquad \text{1-form}$$

$$X \longmapsto D_X \varphi$$

Local coordinates $\quad d\varphi = \sum_i w_i \, du^i$

$$w_i = d\varphi\left(\frac{\partial}{\partial u_i}\right) = D_{\frac{\partial}{\partial u_i}} \varphi = \frac{\partial \varphi}{\partial u_i}$$

So $\quad d\varphi = \sum \frac{\partial \varphi}{\partial u_i} du_i$!

$\boxed{k=1}$ abstract:

$$\omega = 1\text{-form}$$

$^{-1}$

$$d\omega = 2\text{-form}$$

$$d\omega_{z_0} : T_{z_0}M \times T_{z_0}M \longrightarrow \mathbb{R}$$

alternating - bilinear

X, Y vector fields

$$d\omega(X,Y) = D_X\,\omega(Y) - D_Y\,\omega(X) - \omega([X,Y])$$

C^∞
$\mathbb{R}$

Local coordinates $\omega = \sum \omega_i \, du^i$

$$d\omega\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = D_{\frac{\partial}{\partial u_i}}\,\omega\left(\frac{\partial}{\partial u_j}\right) - D_{\frac{\partial}{\partial u_j}}\,\omega\left(\frac{\partial}{\partial u_i}\right) - \omega\left(\left[\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right]\right)$$

$$= \frac{\partial \omega_j}{\partial u_i} - \frac{\partial \omega_i}{\partial u_j}$$

Claim: $d\left(\sum_k' \omega_k \, du^k\right) = \sum_{kl}' \frac{\partial \omega_k}{\partial u_l} du^l \wedge du^k$

Test

$$\left(\sum_{kl}' \frac{\partial \omega_k}{\partial u_l} du^l \wedge du^k\right)\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

$$= \sum_{kl}' \frac{\partial \omega_k}{\partial u_l} du^l\left(\frac{\partial}{\partial u_i}\right) du^k\left(\frac{\partial}{\partial u_j}\right) - \frac{\partial \omega_k}{\partial u_l} du^l\left(\frac{\partial}{\partial u_j}\right) du^k\left(\frac{\partial}{\partial u_i}\right)$$

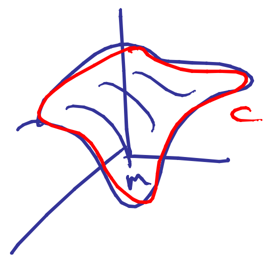
$$= \frac{\partial \omega_j}{\partial u_i} - \frac{\partial \omega_i}{\partial u_j} \qquad \checkmark$$

More generally $\omega = \sum_{i_1 < \cdots < i_k}' \omega_{i_1 \cdots i_k} du_{i_1} \wedge \cdots \wedge du_{i_k}$

$$\Longrightarrow \quad d\omega = \sum_{\substack{i_1 < \cdots < i_k \\ j}}' \frac{\partial \omega_{i_1}}{\partial u_j} du_j \wedge du_{i_1} \wedge \cdots \wedge du_{i_k}$$

Stoke's Thm

$M \hookrightarrow \mathbb{R}^3$ 2-manifold w/ $\partial M = c$



ω = 1-form in $\mathbb{R}^3$ along M

$$\int_M d\omega = \int_{\partial M} \omega$$

(pf) Write

$$\omega = P\,dx + Q\,dy + R\,dz$$

Let $X = (P, Q, R)$ be the correspondg vector field

$$d\omega = \frac{\partial P}{\partial y}\, dy \wedge dx + \frac{\partial P}{\partial z}\, dz \wedge dx$$

$$+ \frac{\partial Q}{\partial x}\, dx \wedge dy + \frac{\partial Q}{\partial z}\, dz \wedge dy$$

$$+ \frac{\partial R}{\partial x}\, dx \wedge dz + \frac{\partial R}{\partial y}\, dy \wedge dz$$

$$= \det \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Final!

$$\int_M d\omega = \int_M \operatorname{curl} X \cdot \vec{dA}$$

$$\Vert \quad \text{Stokes}$$

$$\int_{\partial M} \omega = \int_{\partial M} X \cdot d\vec{s}$$

$\square$

Note: More generally:

M = k-mfld w/ ∂

ω = $(k-1)$-form

$$\int_M d\omega = \int_{\partial M} \omega$$