2 - Prerequisites

Note Title 9/10/2009

$$\mathbb{R}^n = \{\overbrace{(x_1, \dots, x_n)}^{x} \mid x_i \in \mathbb{R}\}$$

vector space $+$, scaler multiplication

Def (pos def)
an inner product on $\mathbb{R}^n$ is a mapping

$$\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

Satisfying

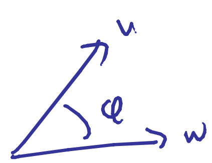
- $\langle v, w \rangle = \langle w, v \rangle$
- $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v, w \rangle + a_2 \langle v, w \rangle$

  $a_i \in \mathbb{R}$ $\quad v, w \in \mathbb{R}^n$
- positive definite

  $\langle v, v \rangle > 0$ for all $v \in \mathbb{R}^n$, $v \neq 0$

$$\|v\| := \sqrt{\langle v, v \rangle}$$

$$\cos \varphi = \frac{\langle x, y \rangle}{\|x\| \|y\|} \qquad \text{(defines } \varphi\text{)}$$



$$\operatorname{dist}(x, y) := \|x - y\|$$

Topology on $\mathbb{R}^n$

A topology on a set X is the notion of "which subsets are open".

Def $x \in \mathbb{R}^n$, $\varepsilon > 0$

$$U_\varepsilon(x) = \{ y \in \mathbb{R}^n \mid \operatorname{dist}(y, x) < \varepsilon \}$$

$U_\varepsilon(x)$

Def: A subset $U \subseteq \mathbb{R}^n$ is open if, for every $x \in U$, there exists $\varepsilon > 0$, so that $U_\varepsilon(x) \subseteq \mathbb{R}^n$

"can talk about limits in $\mathbb{R}^n$"

- e.g. open interval of $\mathbb{R}^1$
- interior of unit ball in $\mathbb{R}^2$

"open" loosely translates to does not contain its boundary

A subset $D \subseteq \mathbb{R}^n$ is closed if D^c is open

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$ "vector valued function"

$$F(x_1, \dots, x_n) = \left(F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_m)\right)$$

<u>Derivative?</u> $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\Updownarrow$$

$$f(x+h) = f(x) + f'(x) \cdot h + e(h)$$

$$\frac{e(h)}{h} \longrightarrow 0$$

<u>Def: F is diff'ble at x</u>

if $\vec{h} \in \mathbb{R}^n$

n × m matrix

f Jacobian

vect valued

f funct of $\vec{h}$

$$F(x + \vec{h}) = F(x) + (J_x F)\vec{h} + e(\vec{h})$$

s.t.

$$\lim_{\vec{h} \to 0} \frac{e(\vec{h})}{\|\vec{h}\|} \longrightarrow 0$$

Thm If F is differentiable at $x \Rightarrow \dfrac{\partial F_i}{\partial x_j}$ exists

and
$$J_x F = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

Jacobian is "linear approximation to F at x"

Converse not true

could have $\dfrac{\partial F_i}{\partial x_j}$ exist but

F not diff'ble at x

However If $\dfrac{\partial F_i}{\partial x_j}$ exist and are continuous

$\Longrightarrow F$ diff'ble

More generally F C^r means $\dfrac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}}$ exist and are continuous

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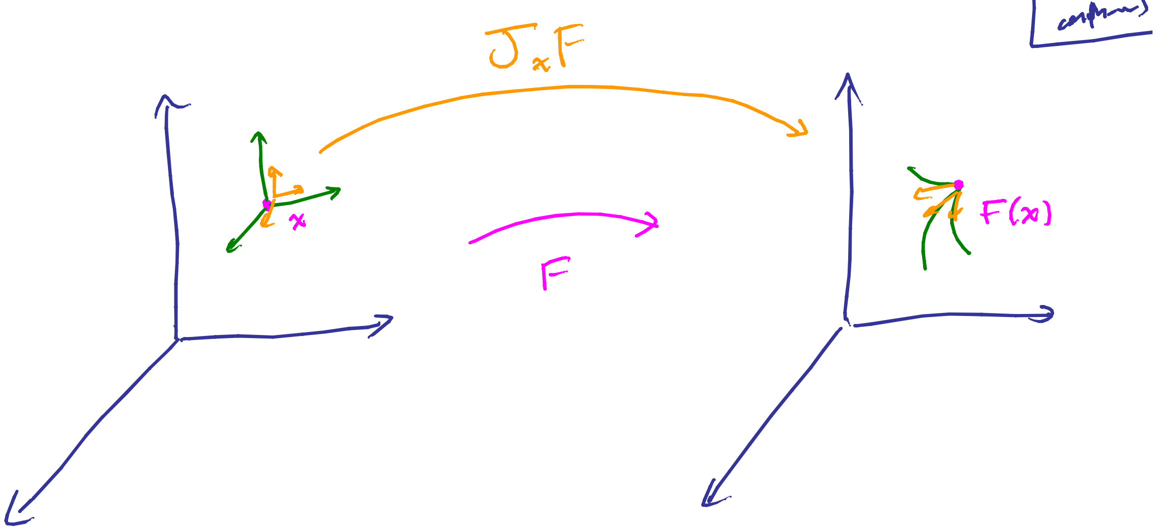


$J_x F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

is a linear transformation

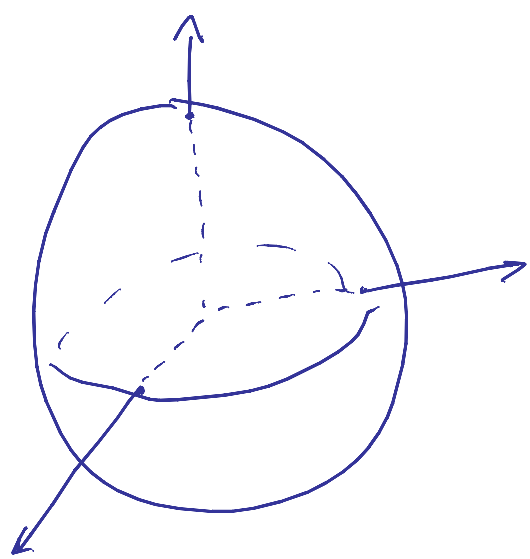
Def

$J_x F$ injective $\forall x \Rightarrow F$ is an immersion

$J_x F$ surjective $\forall x \Rightarrow F$ is a submersion

Implicit function thm $\qquad S^2 \subseteq \mathbb{R}^3$

problem



$$S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$$

" S^2 is defined by $\qquad S^2 = F^{-1}(0)$

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto x^2+y^2+z^2-1$$ "

Know S^2 admits parametrization

$$x(\theta,\varphi)$$
$$y(\theta,\varphi)$$
$$z(\theta,\varphi)$$

Can we deduce this USING only F?

$z(x,y)=\sqrt{1-x^2-y^2}$

$$\begin{tikzcd}
S^2 \arrow[r, hook] \arrow[d, equal] & \mathbb{R}^3 \arrow[r, "F"'] & \mathbb{R} \\
F^{-1}(0) & \text{3-dim'l} & \text{1-dim'l} \\
\text{2-dim'l} & &
\end{tikzcd}$$

Consider equator $S^1 \hookrightarrow \mathbb{R}^3$

$$\begin{cases} x \\ y(x)=\sqrt{1-x^2} \\ z(x)=0 \end{cases}$$

$$\left.\begin{aligned} x^2+y^2&=1 \\ z&=0 \end{aligned}\right\} \text{2-equations}$$

$$G : (x,y,z) \longmapsto (x^2+y^2-1,\ z)$$

$$\begin{tikzcd}
S^1 \arrow[r] & \mathbb{R}^3 \arrow[r, "G"'] & \mathbb{R}^2 \\
1 & 3 & 2
\end{tikzcd}$$

In general

$$\begin{array}{ccccc} X & \hookrightarrow & \mathbb{R}^n & \xrightarrow[F]{} & \mathbb{R}^m \\ \parallel & & & & \\ F^{-1}(0) & & & & \end{array}$$

$(n-m)$-dim'l

Q: when can we parametrize X w/ $k = n-m$ different params.

<u>Implicit Function Thm</u>

$$\begin{array}{ccc} \mathbb{R}^n & = & \mathbb{R}^k \times \mathbb{R}^m \\ \cup & & \cup \\ U & = & U_1 \times U_2 \end{array}$$

U_i open

$$(x_1, \ldots, x_k, y_1, \ldots, y_m) = (x, y)$$

$$F : U \longrightarrow \mathbb{R}^m \qquad C^\infty$$

Suppose $(a,b) \in U$

satisfies

- $F(a,b) = 0$
- $\left[\dfrac{\partial F_i}{\partial y_j}\bigg|_{(a,b)} \right]$ invertible

$\Rightarrow \quad \exists \quad \underset{\underset{(a,b)}{\Psi}}{V_1 \times V_2} \subset U_1 \times U_2$

$\exists \quad C^\infty \quad g : V_1 \longrightarrow V_2$

$$\left.\begin{array}{l} y_1 = y_1(x_1, \ldots, x_k) \\ \vdots \qquad\qquad \vdots \\ y_m = y_m(x_1, \ldots x_k) \end{array}\right\} y(x)$$

s.t. $F(x, y(x)) = 0 \quad$ for all $x \in V_1$

In other words

$(x_1, \ldots, x_n) \longmapsto (x_1, \ldots, x_k, \; y_1(x_1, \ldots x_k), \ldots, y_m(x_1 \ldots x_k))$

is a parametrization

of $X = F^{-1}(0)$

$\underline{\underline{\text{near}}}$ (a,b)

(e.g. S^1, S^2)

(Inverse function thm) $\quad f : \underset{\underset{u_0}{\Psi}}{U} \longrightarrow \mathbb{R}^n \quad C^\infty$

$U \subseteq \mathbb{R}^n \qquad Jf|_{u_0}$ invertible

$\Rightarrow \exists \; V \subseteq U \quad f : V \longrightarrow f(V)$ diffeomorphism

Def A k-dim'l submanifold (of class C^∞)

$$M \subseteq \mathbb{R}^n$$

satisfies: for every point $p \in M$, there exists $U \subseteq \mathbb{R}^n$ open nbhd, $U \ni p$

and $F : U \longrightarrow \mathbb{R}^{n-k}$

with $\operatorname{rank} J_x F = n-k$ [maximal rank]

for all $x \in M \cap U$

(i.e. $J_x F$ is surjective)

so that $M = F^{-1}(0)$

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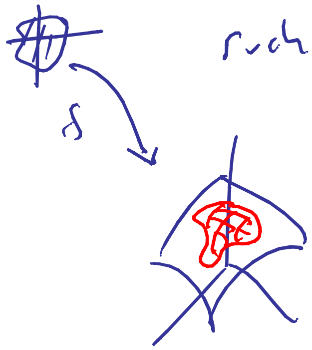
e.g. S^2

Def: Let $M \subset \mathbb{R}^n$ be a k-dim'l submfld.

A local parametrization is

a mapping

$$\begin{array}{ccc} V & \xrightarrow{f} & M \\ \cap & & \cap \\ \mathbb{R}^k & & \mathbb{R}^n \end{array}$$



such that $\operatorname{rank}(J_x f) = k$

for all $x \in V$

i.e. $J_x f$ is <u>injective</u>

"Immersion"

<u>Prop</u> $M \subset \mathbb{R}^n$ k-dim'l submfld

$p \in M$, $\exists$ local parametrization

$$V \xrightarrow{f} M$$

$$p \in f(V)$$

(pf) Since submfld

$$\exists \ p \in U \xrightarrow{F} \mathbb{R}^{n-k} \quad \text{full rank}$$

$$U \subset \mathbb{R}^n$$

$z_1, \dots, z_n$

$$M \cap U = F^{-1}(0)$$

$$\left[\frac{\partial F_i}{\partial z_j}\bigg|_p \right] \quad \text{full rank}$$

$\Longrightarrow$ reordering $z_1, \dots, z_n \rightsquigarrow (x_1, \dots, x_k, y_1, \dots, y_{n-k})$

$$\left[\frac{\partial F_i}{\partial y_j}\Big|_p \right] \text{ invertible}$$

<u>IFT</u> $\Longrightarrow$ $\exists \quad V_1 \times V_2 \subseteq U$

$$y_i = y_i(x_1, \dots, x_k)$$

$$(x_1, \dots, x_k) \longmapsto (x_1, \dots, x_k, y_1(x_1, \dots, x_k), \dots, y_{n-k}(x_1, \dots, x_k))$$

$$\begin{array}{ccc} V_1 & \xrightarrow{\ f\ } & M \subseteq \mathbb{R}^n \\ \cap & & \\ \mathbb{R}^k & & \end{array}$$

Just need to check df_z is an immersion.

$$Jf = \left[\frac{\partial f_i}{\partial x_j} \right] = \begin{array}{c} \overbrace{\hphantom{1 \quad 0 \quad 0 \quad \cdots \quad 0}}^{k} \\ \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{\partial y_1}{\partial x_1} & & \cdots & & \frac{\partial y_1}{\partial x_k} \\ \vdots & & & & \vdots \\ \frac{\partial y_{n-k}}{\partial x_1} & & \cdots & & \frac{\partial y_{n-k}}{\partial x_k} \end{bmatrix} \end{array} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \text{rank} = k \\ \\ \\ \\ \end{array}$$

✓

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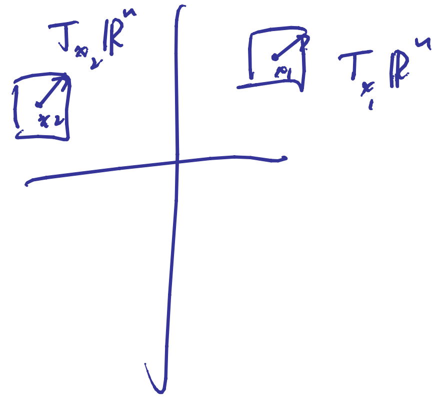
e.g. open subsets of $\mathbb{R}^n$

Tangent spaces

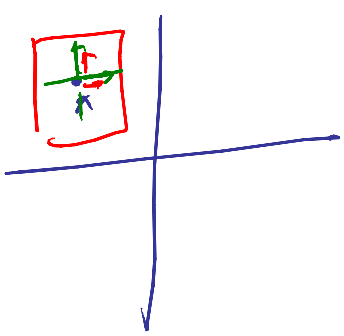
$x \in \mathbb{R}^n$

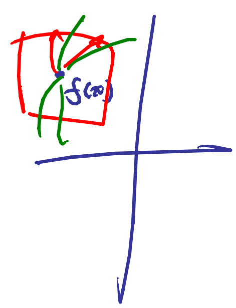
$$T_x\mathbb{R}^n = \mathbb{R}^n$$

vector space



$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$





$x \in \mathbb{R}^n$

$$Df\big|_x : T_x\mathbb{R}^n \longrightarrow T_{f(x)}\mathbb{R}^m$$

linear transformation

$$\vec{v} \longmapsto J_x f \cdot \vec{v}$$

different one for each x

Tangent space of mfld

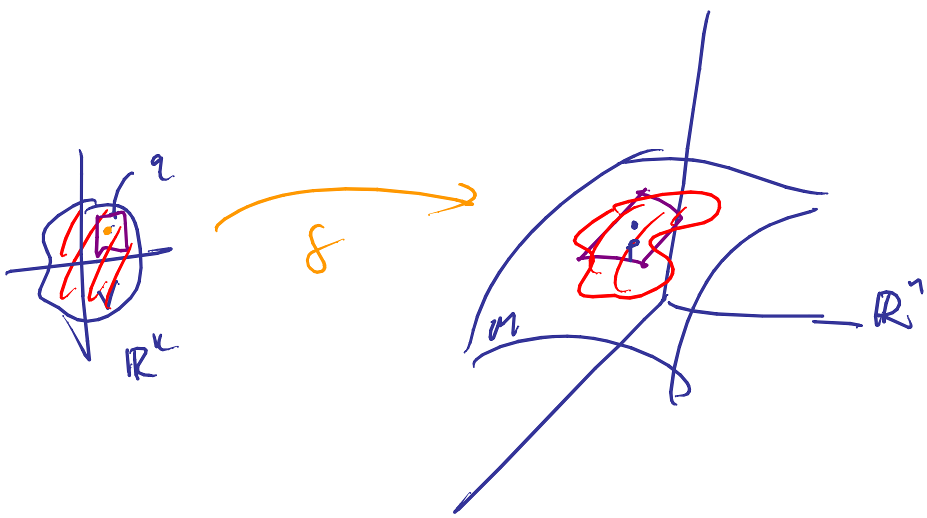
$M \subseteq \mathbb{R}^n$ k-dim'l submfld

$p \in M$

Choose a local parametrizn of M in nbhd of p

$$f : V \longrightarrow M$$
$$V \subseteq \mathbb{R}^k$$

Ask: how to define tangent space



$$T_q \mathbb{R}^k \xrightarrow{Df|_q} T_p \mathbb{R}^n$$

$T_q M$ is defined to be image$(Df|_q)$

- k-dim'l
- independent of local parametrization,

Normal Space ($n-k$-dim'l)

$$T_p\mathbb{R}^n = T_pM \oplus \perp_p M$$