20 Gauss - Bonnet

Note Title 12/1/2009

<u>Thm</u> (Gauss-Bonnet) local version

$$\mathbb{R}^2 \supseteq U \xrightarrow{f} M \subseteq \mathbb{R}^3$$

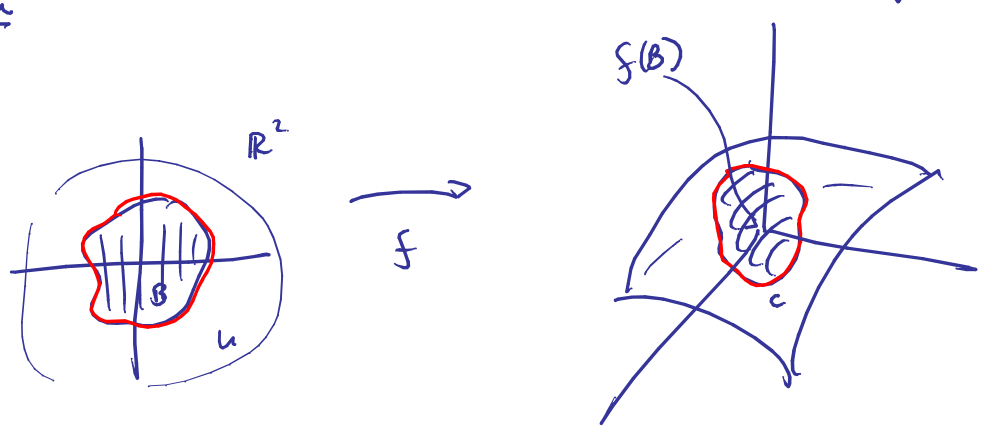
$U \supseteq B$ ← diffeomorphic to disk

$\partial f(B) = c$

$$\int_c \kappa_g \, ds + \int_{f(B)} K \, dA = 2\pi$$

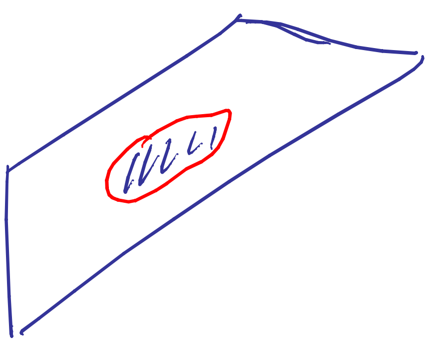
(c oriented w/ B on left)

<u>Picture</u>



<u>e.g.</u> M = plane $K = 0$

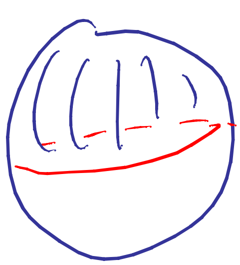
$$\kappa_g = \kappa$$



$$\int_c \kappa \, ds = 2\pi$$

(thm of turning tangents)

<u>e.g.</u>

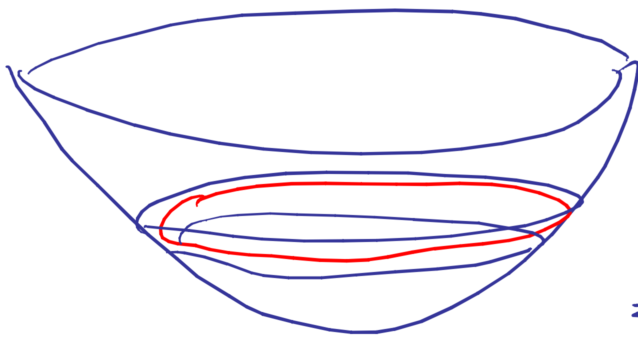


$f(B)$ = northern hemisphere

$\kappa_g = 0$, $K = 1$

$$\int_{f(B)} dA = \frac{4\pi}{2} = 2\pi$$

$K > 0$

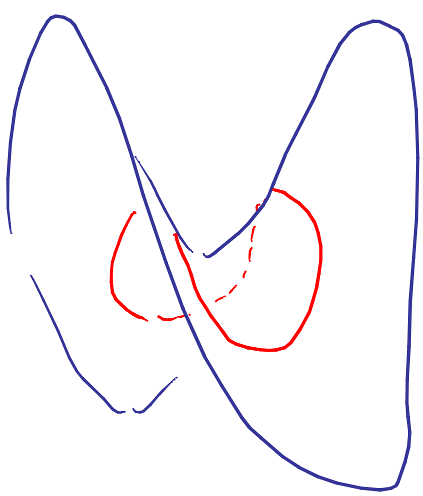


driving around curved racetrack

$\Rightarrow$ need to turn the steering wheel less

$\Rightarrow \int k_g < 2\pi$

$K < 0$



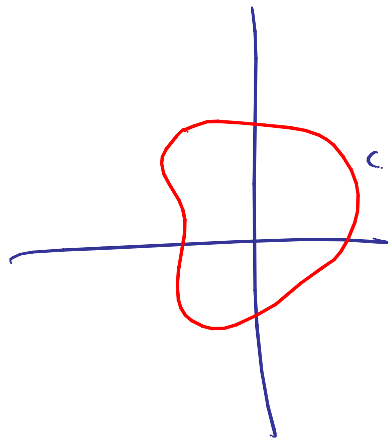
need to turn steering wheel <u>more</u>!

$\int k_g > 2\pi$

<u>Recall</u> thm of turning tangents

$$\int_c \kappa = 2\pi$$

simple closed curve c



$$e_1 = c'$$

$$e_1' = \kappa e_2$$

$$e_2' = -\kappa e_1$$

$$e_1 = (\cos\theta(t), \sin\theta(t))$$

$$e_2 = (-\sin\theta(t), \cos\theta(t))$$

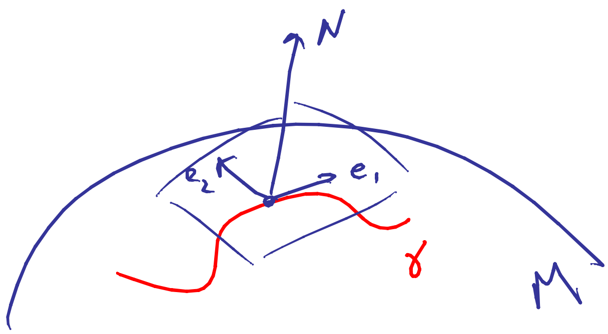
Compute $\quad e_1' = \left((-\sin\theta)\theta', (\cos\theta)\theta'\right)$

$$= \theta' e_2$$

$$\Rightarrow \kappa = \theta'$$

$$\int \kappa \, ds = \int_a^b \frac{d\theta}{dt} \, dt = \theta(b) - \theta(a) = 2\pi$$

<u>Geodesic</u> <u>curvature</u> M orientable mfld



choose $e_2(t)$ s.t. $(e_1(t), e_2(t), N(\gamma(t)))$

is an positively oriented orthonormal basis

of $T_{\gamma(t)}\mathbb{R}^3$

$$\langle e_1, e_1 \rangle = 1$$

$$\Rightarrow \langle e_1', e_1 \rangle = 0$$

$$\Rightarrow D_{e_1} e_1 = e_1' = \kappa_\nu N + \kappa_g e_2$$

$$\Rightarrow \nabla_{e_1} e_1 = \kappa_g e_2$$

$\kappa_g > 0 \Rightarrow$ turning left

$\kappa_g < 0 \Rightarrow$ turning right (in fact

$$\nabla_{e_1} e_1 = \kappa_g e_2$$

$$\nabla_{e_1} e_2 = -\kappa_g e_1$$

"frenet" eqns)

Let (X_1, X_2, N) be orthonormal positively oriented basis of vector fields in $\mathbb{R}^3$ along M

$X_1(z), X_2(z) \in T_z M$

Write:

$$e_1(t) = \cos(\theta(t)) X_1 + \sin(\theta(t)) X_2$$

$$\Rightarrow \quad e_2(t) = -\sin(\theta(t)) X_1 + \cos(\theta(t)) X_2$$

$$\nabla_{e_1} e_1 = -\sin(\theta)\theta' X_1 + \cos(\theta)\theta' X_2$$

$$\parallel$$

$$\kappa_g e_2 \qquad + \underbrace{\cos\theta\left(\nabla_{e_1} X_1\right) + \sin\theta\left(\nabla_{e_1} X_2\right)}$$

$$\nabla_{e_1} X_1 = \omega_1^2(e_1) X_2$$

$$-(\cos\theta)\, \omega_2^1(e_1) X_2$$

$$\nabla_{e_2} X_1 = \omega_2^1(e_1) X_1$$

$$+ (\sin\theta)\, \omega_2^1(e_1) X_1$$

$$= \left(\theta' - \omega^1_2(e_1)\right) e_2$$

$$\Rightarrow \quad \kappa_g = \theta' - \omega^1_2(e_1)$$

$$\Rightarrow \quad \kappa_g + \omega^1_2(e_1) = \theta'$$

$$\Rightarrow \quad \int_c \kappa_g \, ds + \int_{\partial f(B)} \omega^1_2 = \int_a^b \theta' \, dt$$

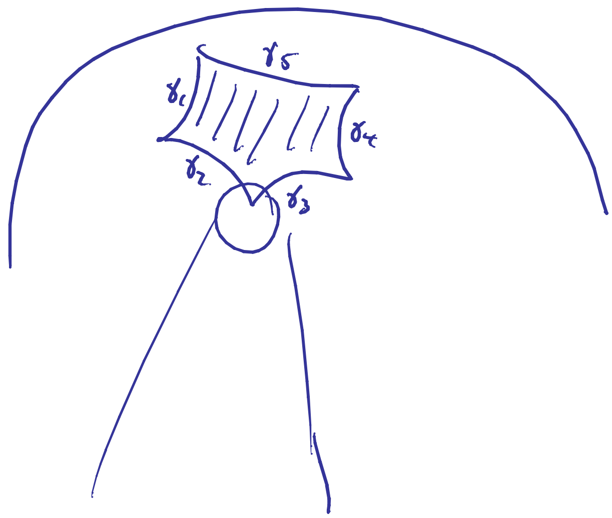
$$\parallel \qquad\qquad = \theta(b) - \theta(a) = 2\pi$$

$$\int_c \kappa_g \, ds + \underbrace{\int_{f(B)} d\omega^1_2}_{\parallel}$$

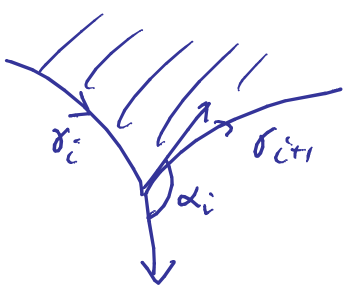
$$\int_{f(B)} K \, \omega^1 \wedge \omega^2 = \int_{f(B)} K \, dA \qquad \Box$$

dual to ON basis

Gauss-Bonnet - local polygon version



$f(B) \approx$ Disk
homeomorphic



$\gamma_i : [a_i, b_i] \longrightarrow M$

$\theta_i(t)$

$\alpha_i = \theta_{i+1}(a_i) - \theta_i(b_i)$

$\alpha_k = \theta_1(a_1) - \theta_k(b_k) + 2\pi$

Then (same notation)

$$\int K_g \, ds + \int K \, dA = \sum_i \int_{a_i}^{b_i} \theta_i' \, ds$$

$$= \left(\theta_1(b_1) - \theta_1(a_1)\right) + \cdots + \left(\theta_k(b_k) - \theta_k(a_k)\right)$$

$$= \left(\theta_1(b_1) - \theta_2(a_2)\right) + \left(\theta_2(b_2) - \theta_3(a_3)\right) + \cdots + \left(\theta_{k-1}(b_{k-1}) - \theta_k(a_k)\right)$$

$$+ \left(\theta_k(b_k) - \theta_1(a_1)\right)$$

$$= \left(\sum_{i=1}^{k-1} -\alpha_i\right) - \alpha_k + 2\pi$$

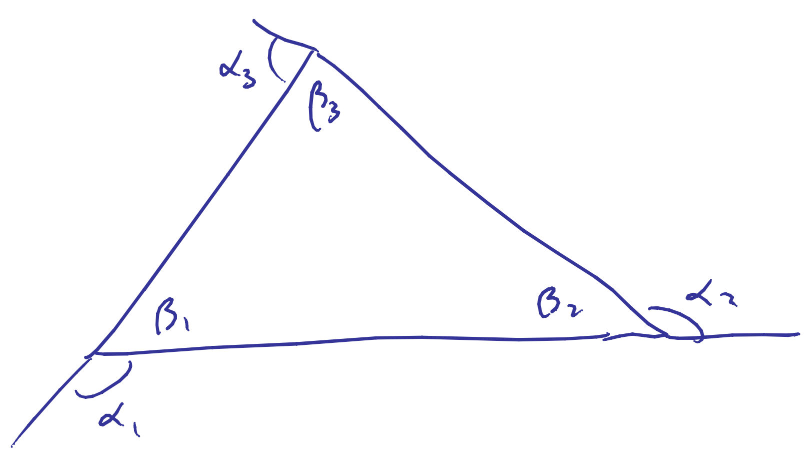
Thm

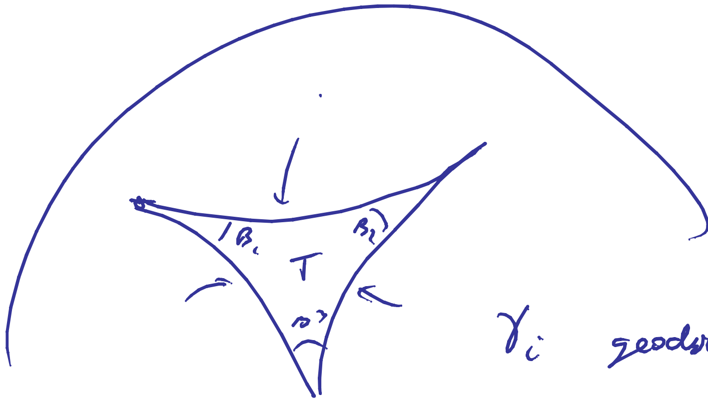
So
$$\boxed{\int_{\gamma_i} \kappa_g\, ds + \sum_{i=1}^{k} \alpha_i + \int_{f(B)} K\, dA = 2\pi}$$

Special case!

geodesic Δ's

$\beta_i = \pi - \alpha_i$



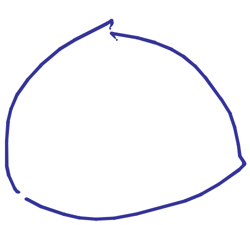


γ_i geodesics

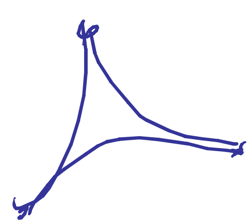
$$\int_T K\,dA + \underbrace{\alpha_1 + \alpha_2 + \alpha_3}_{3\pi - \sum \beta_i} = 2\pi$$

$$\Longrightarrow \quad \beta_1 + \beta_2 + \beta_3 = \pi + \int_T K\,dA \quad \left(+ \int k_g \right)$$

e.g. $\quad K > 0 \qquad\qquad\qquad K < 0$



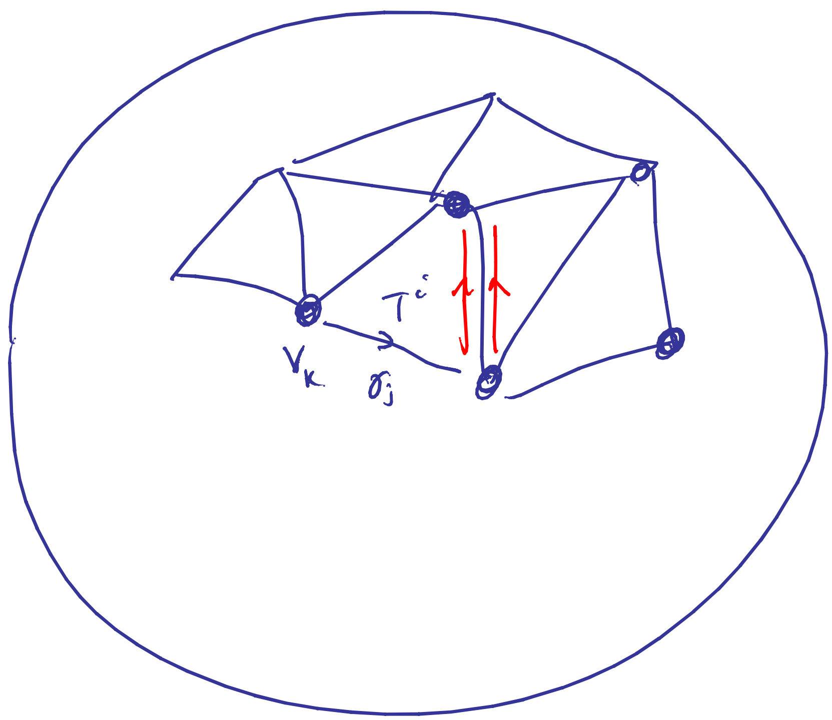
FAT TRIANGLES



THIN TRIANGLES

Gauss - Bonnet - Thm

M = closed orientable surface



T^i has interior angles $\beta_1^i, \beta_2^i, \beta_3^i$

$$\int_M K \, dA = \sum_{i,l} \beta_l^i$$

$$+ F \cdot \pi + \int_{\gamma_i} \kappa_g + \int_{-\gamma_i} \kappa_g$$

$$\int_M K\,dA = -F\pi + \sum_{k} \sum_{\substack{\beta_l^i \text{ touches} \\ V_k}} \beta_l^i$$

$$= -F\pi + V \cdot 2\pi$$

But $$E = \frac{3F}{2}$$

every face has 3 edges, but this double counts them

So $$2\pi\chi(M) = 2\pi(F - E + V)$$

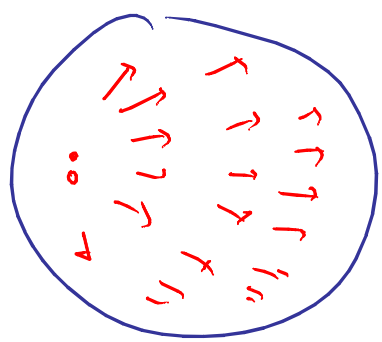
$$= 2\pi\left(F - \frac{3F}{2} + V\right)$$

$$= 2\pi\left(\frac{-F}{2} + V\right)$$

$$= -F\pi + 2\pi V \qquad \square$$

Application: "Hairy ball thm"

Thm: There does not exist a nowhere vanishing vector field on S^2



vector field must vanish somewhere

"grow hair on a ball and try to comb it"

(Pf) Suppose X is a nowhere vanishing vector field on S^2

Write $X_1 = \frac{X}{\|X\|}$

$$X_3 := N$$

$$X_2 := X_3 \times X_1$$

Then X_1, X_2, X_3 forms an orthonormal frame.

Form $\omega^1, \omega^2, \omega^3, \quad \omega^i_j$ etc.

$$d\omega^1_2 = K\, \omega^1 \wedge \omega^2 = K dA$$

$$0 = \int_{\substack{\partial S^2 \\ \| \\ \emptyset}} \omega^1_2 = \int_{S^2} d\omega^1_2 = \int_{S^2} K\, dA = 2\pi\, \chi(S^2) = 4\pi$$

$\rightarrow\leftarrow$

$\square$