21 - Riemannian Manifolds

Note Title 12/3/2009

Final:

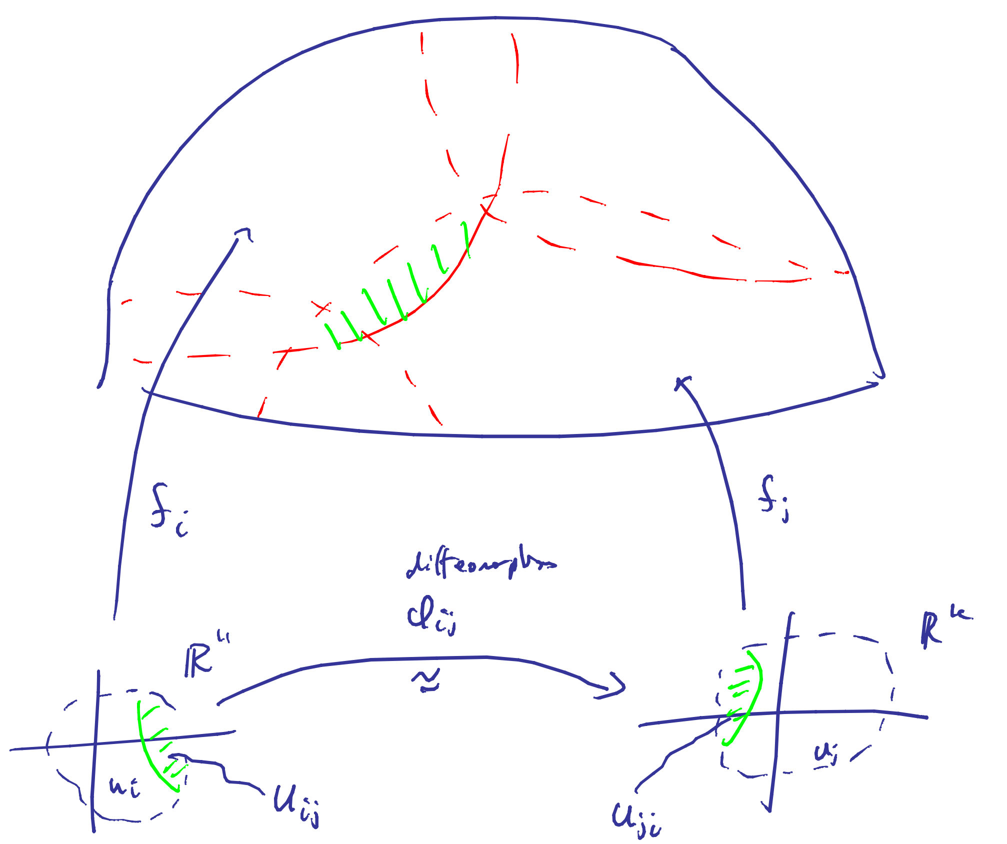
Let $M \subseteq \mathbb{R}^n$ be a k-submfld.

Recall that M admits an <u>ATLAS</u>

$$\left\{ f_i : \underset{\substack{\cap \text{ open} \\ \mathbb{R}^k}}{U_i} \longrightarrow M_i \right\}$$

local coordinate charts

s.t. $M = \bigcup U_i$



$\varphi_{ij} = f_j^{-1} f_i$

"just specify coordinate patches & gluing data"

Def: A $\underline{k\text{-manifold}}$ M is a collection $\{U_i, U_{ij}, \varphi_{ij}\}$

Satisfying

(1) $U_i \subseteq \mathbb{R}^k$ open

(2) $U_{ij} \subseteq U_i$ open $i \neq j$ $(U_{ii} = U_i)$

(3) $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ diffeomorphism

$(\varphi_{ii} = Id)$

(4) $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$

consequence

$(\varphi_{ij} = \varphi_{ji}^{-1})$

(Motivation $f_k^{-1} f_j f_j^{-1} f_i = f_k^{-1} f_i$)

disjoint union

$M := \coprod_i U_i / \sim$ $x \in U_i$ $y \in U_j$

$x \sim y$ if $x \in U_{ij}$

$y \in U_{ji}$

$\varphi_{ij}(x) = y$

Minor topological technical details

Hausdorff condition:

$$U_i \xhookrightarrow[f_i]{} M$$

We say $U \subseteq M$ is open

if $f_i^{-1}(U)$ open for all i

We require

$z_0 \neq z_1 \in M$, then $\exists$ $\overset{z_0}{\overset{\cap}{V_0}}, \overset{z_1}{\overset{\cap}{V_1}} \underset{\text{open}}{\subset} M$

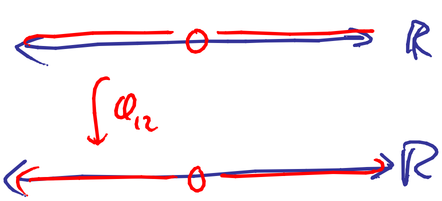
so that $V_0 \cap V_1 = \emptyset$

Non-example "double point"

$U_1 = \mathbb{R}$

φ_{12}

$U_2 = \mathbb{R}$

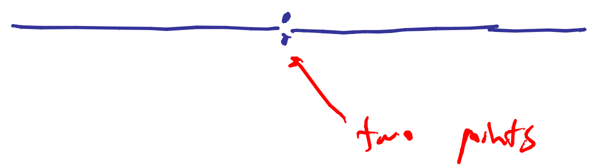


$U_{12} = \mathbb{R} - \{0\}$

$U_{21} = \mathbb{R} - \{0\}$

$\varphi_{12} = \mathrm{Id} : U_{12} \to U_{21}$

glue: get



Para-compactness condition

M is paracompact as a topological space

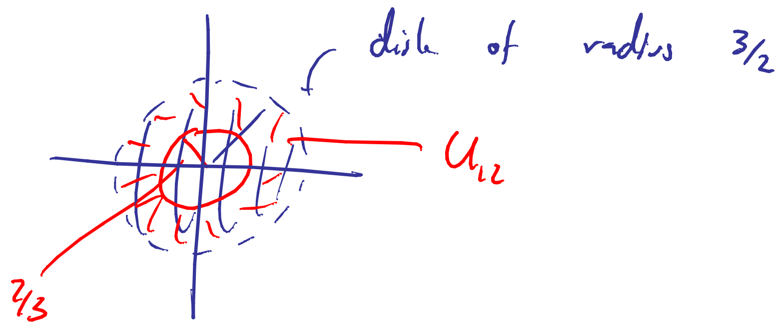
(e.g. if $\{U_i\}$ is countable, then M is paracompact)

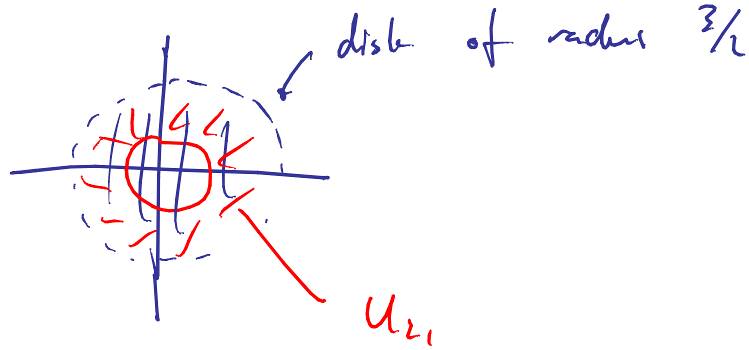
Non-example

take uncountably many intervals, and glue them together

get "long line": not paracompact!

example S^2 as an abstract manifold

$U_1 =$ 

$U_2 =$ 

$$\varphi_{12} : U_{12} \longrightarrow U_{21}$$

$$(x,y) \longmapsto \left(\frac{x}{x^2+y^2} , \frac{y}{x^2+y^2} \right)$$

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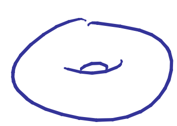
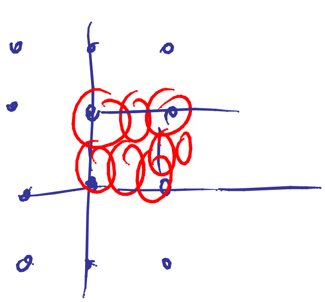
example torus  $= \mathbb{R}^2/\sim$



cover w/ open disk of radius < 1

$(x,y) \sim (x+k, y+\ell)$

$(k,\ell) \in \mathbb{Z}$

Tangent space

$$\begin{tikzcd} U \arrow[r, "f"] & M \\ \bar{U} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

$U = U_s$

$\bar{U} = U_t$

$z \in f(U)$

$U \subseteq \mathbb{R}^k$ coordinates

$u_1 \dots u_k$

$T_z M$ = $\mathbb{R}$-vector space w/ basis

$$\left\{ \frac{\partial}{\partial u_i} \right\}$$

$z \in f(U) \cap \bar{f}(\bar{U})$

$T_z M$ has two different bases

$$\frac{\partial}{\partial u_i}, \quad \frac{\partial}{\partial \bar{u}_i}$$

get

$$\begin{array}{cccc} \varphi_{s,t} : & U_{s,t} & \longrightarrow & U_{t,s} \\ & \cap & & \cap \\ & U_s & & U_t \end{array}$$

$$\bar{u}_i = \bar{u}_i(u_1, \dots, u_k)$$

$$\frac{\partial}{\partial u_i} = \sum_j \frac{\partial \bar{u}_j}{\partial u_i} \frac{\partial}{\partial \bar{u}_j}$$

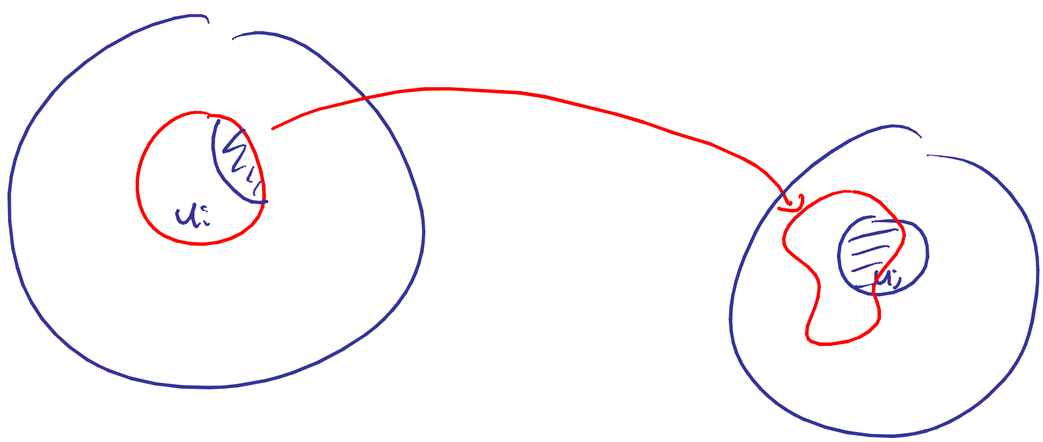
$$X \in T_z M \Rightarrow X = \sum X^i \frac{\partial}{\partial u_i}$$

<u>Smooth</u> f<u>unctions</u> M^m, N^n manifolds

A mapping $M \xrightarrow[\varphi]{} N$ is smooth

$$\begin{array}{ccccccc}
U_i \cap f_i^{-1}(\varphi^{-1}(U_j)) & \xrightarrow{f_j^{-1} \circ \varphi \circ f_i} & & & & & \\
\cap & & & & & & \\
U_i & \xrightarrow[f_i]{} & M & \xrightarrow[\varphi]{} & N & \xleftarrow[f_j]{} & U_j \\
\cap\!| & & & & & & \cap\!| \\
\mathbb{R}^m & & & & & & \mathbb{R}^n
\end{array}$$

smooth



Directional derivatives

$$X \in T_z M \qquad X = \sum X^i \frac{\partial}{\partial u_i}$$

$$\begin{tikzcd} & M \arrow[r] & \mathbb{R} \\ & U_i \arrow[u, "f_i"] \arrow[ur, "\varphi(u_1, \dots, u_n)"'] & \end{tikzcd}$$

$\varphi : M \to \mathbb{R}$ smooth

$$D_X \varphi := \sum_i X^i \frac{\partial \varphi}{\partial u_i}\Big|_z$$

"$X(\varphi)$"

Principle

consider

$$D_X(-) : \left\{ M \xrightarrow{\text{smooth}} \mathbb{R} \right\} \longrightarrow \mathbb{R}$$

Let $\varphi(u_1, \dots, u_n) = u_j$ "call this φ_{u_j}"

$$D_X \varphi = \sum X^i \frac{\partial u_j}{\partial u_i} = X^j$$

So given $D_X(-)$, can read off X^j

$$\boxed{\text{Directional derivative determines } X}$$

Jacobean

$$M^m \xrightarrow{F} N^n$$

$$DF\big|_z : T_z M \longrightarrow T_z N$$

two definitions

(i) local coordinates

choose local coordinates

$(u_1, \dots, u_m)$ of M $\qquad v_i = v_i(u_1, \dots, u_m)$

$(v_1, \dots, v_n)$ of N

$$DF\left(\sum_i X^i \frac{\partial}{\partial u_i}\right) = \sum_{ij} X^i \frac{\partial v_j}{\partial u_i} \frac{\partial}{\partial v_j}$$

Alternative $\quad X \in T_z M$

$$D_{DF(X)} \varphi = D_X (\varphi \circ F)$$

$$M \xrightarrow{F} N \xrightarrow{\varphi} R$$

Lie Bracket

Def X, Y vector fields on M

Define $[X,Y]$ to be the unique vector field satisfying

$$D_{[X,Y]}\varphi = D_X(D_Y\varphi) - D_Y(D_X\varphi)$$

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local coordinates

$$X = \sum_i X^i \frac{\partial}{\partial u_i}$$

$$Y = \sum_i Y^i \frac{\partial}{\partial u_i}$$

$$D_{[X,Y]} u_i = D_X Y^i - D_Y X^i$$

$$= \sum_j \left(X^j \frac{\partial Y^i}{\partial u_j} - Y^j \frac{\partial X^i}{\partial u_j} \right)$$

Note $[X,Y] = -[Y,X]$

$$[X, \varphi Y] = (D_X\varphi)Y + \varphi[X,Y]$$

Connections

A connection ∇ on M

- $X \in T_z M$ tangent vector
- Y vector field on M

gives $\nabla_X Y \in T_z M$

s.t. (linear in X)

(i) $\nabla_{c_1 X_1 + c_2 X_2} Y = c_1 \nabla_{X_1} Y + c_2 \nabla_{X_2} Y$

(ii) $\nabla_X (Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

(iii) $\nabla_X (\varphi Y) = (D_X \varphi) \cdot Y + \varphi \nabla_X Y$

(iv) If X is a vector field

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Local coordinates $\qquad \Gamma_{ij}^k = \Gamma_{ij}^k(u_1, \ldots, u_n)$

$$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u_k}$$

$$\nabla_{\frac{\partial}{\partial u_i}} \sum_j X^j \frac{\partial}{\partial u_j} = \sum_j \frac{\partial X^j}{\partial u_i} \frac{\partial}{\partial u_j} + \sum_{j,k} X^j \Gamma_{ij}^k \frac{\partial}{\partial u_k}$$

$$= \sum_k \left(\frac{\partial X^k}{\partial u_i} + \sum_j X^j \Gamma_{ij}^k \right) \frac{\partial}{\partial u_k}$$

(iv) $$\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \nabla_{\frac{\partial}{\partial u_j}} \frac{\partial}{\partial u_i}$$

$$\Longrightarrow \boxed{\Gamma_{ij}^k = \Gamma_{ji}^k} \quad \#$$

for any random collection of functions $\{\Gamma_{ij}^k\}_{ijk}$

satisfying (#), get

connection. To get a preferred connection, you need...

Metric !

A Riemannian metric on M is a tensor of type $(0,2)$ on M

i.e. $\langle -,- \rangle_z : T_zM \times T_zM \longrightarrow \mathbb{R}$ $\forall z \in M$

- symmetric
- bilinear
- positive definite

locally $g_{ij}(u_1, \ldots, u_n) = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right\rangle$

$$\langle X, Y \rangle = \sum_{ij} X^i Y^j g_{ij}$$

We say $(M, \langle -,- \rangle)$ is a Riemannian Manifold

We call a connection ∇ a Riemannian (or Levi-Civita) if

$$D_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_Y Z \rangle$$

Thm Let $(M, \langle -,- \rangle)$ be a Riemannian mfld, then $\exists !$ Riemannian connection ∇

(pf)

$$D_{\frac{\partial}{\partial u_i}} \left\langle \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k} \right\rangle$$

$$= \Gamma_{ij,k} - \Gamma_{ik,j}$$

Then

$$-D_{\frac{\partial}{\partial u_k}}\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right\rangle + D_{\frac{\partial}{\partial u_i}}\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_k}\right\rangle + D_{\frac{\partial}{\partial u_j}}\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_k}\right\rangle$$

$\parallel$ "old formula"

$$2\Gamma_{ij,k}$$

Conversely given $\Gamma_{ij,k}$ as above

Check

$$\Gamma_{ij,k} - \Gamma_{ik,j} = D_{\frac{\partial}{\partial u_i}}\left\langle \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right\rangle$$

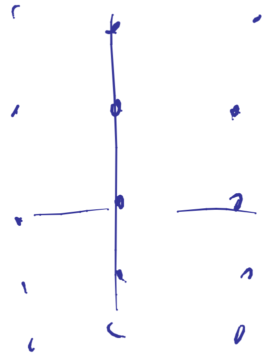
Geodesics

$$\nabla_{\dot{c}}\, \dot{c} \equiv 0$$

Parallel vect fields

$$\nabla_{\dot{c}}\, Y \equiv 0$$

example: flat torus



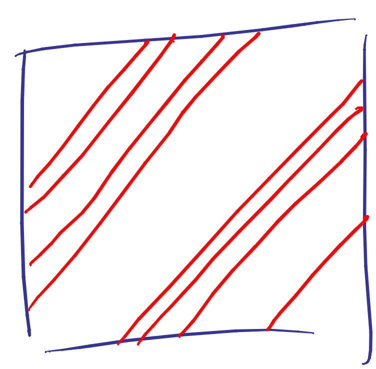
$U \subseteq \mathbb{R}^2$

(x, y)

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Gamma_{ij}^{k} \equiv 0$$

Geodesics



Curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$$
$$- \nabla_{[X,Y]} Z$$

Check it only depends on $X_{t_0}, Y_{t_0}, Z_{t_0}$

$$R(\varphi X, Y) Z = \varphi R(X,Y) Z$$

etc. . . .