22 - Sectional Curvature

Note Title 12/10/2009

$$\mathcal{K}(X,Y) = \langle R(X,Y)Y, X\rangle$$

σ = plane $\subset T_zM$

X, Y = orthonormal vectors which span σ

$K_\sigma := \mathcal{K}(X,Y)$ (independent of choice of X, Y)

(e.g. M = surface, $\sigma = T_zM$, $K_\sigma = K$)

<u>Thm 6.5</u>

$$\langle R(X,Y)Z, W\rangle = \left(\begin{array}{c}\text{some horrible}\\ \text{formula in-lvs,}\\ \text{only } \mathcal{K}\end{array}\right)$$

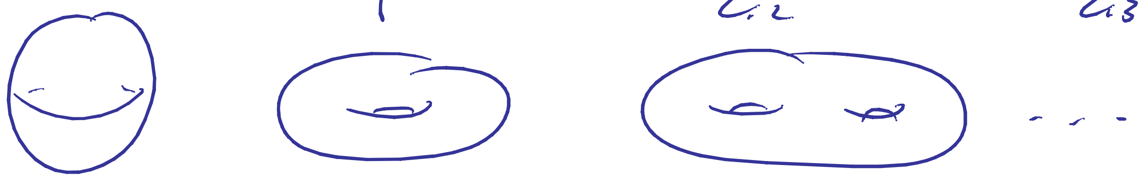
$\Rightarrow$ $\mathcal{K}$ determines R

M has constant sectional curvature if $K_\sigma \equiv K$

Thm any two Riem mflds w/ same constant sectional curvature are locally isometric.

Back to surfaces:

Classification of orientable surfaces



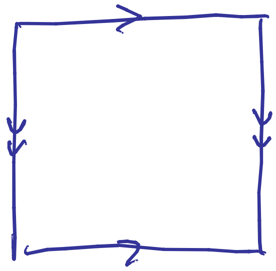
	S^2	T^2	Σ_2	Σ_3	...
$g =$	0	1	2	3	
$\chi =$	2	0	-2	-4	

We have already know S^2 admits a metric w/ $K \equiv 1$ $\left(\int K = 4\pi\right)$

T^2 admits metric w/ $K \equiv 0$ $\left(\int K = 0\right)$

Thm For $g \geq 2$, Σ_g admits metric with $K \equiv -1$ $\left(\int K = 2\pi(2-2g) < 0\right)$

First step: polygonal representations of Σ_g

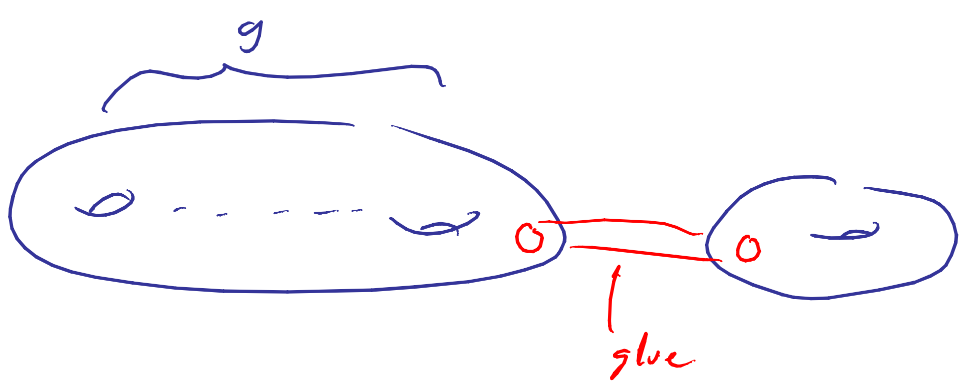
$T^2 =$ 

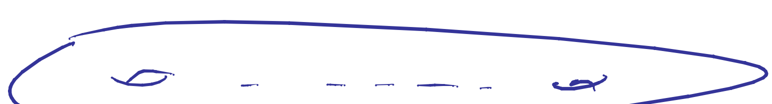
Connect sum

Σ, Σ' two surfaces

$\Sigma \# \Sigma'$: cut out disk in Σ and Σ',
glue along ∂

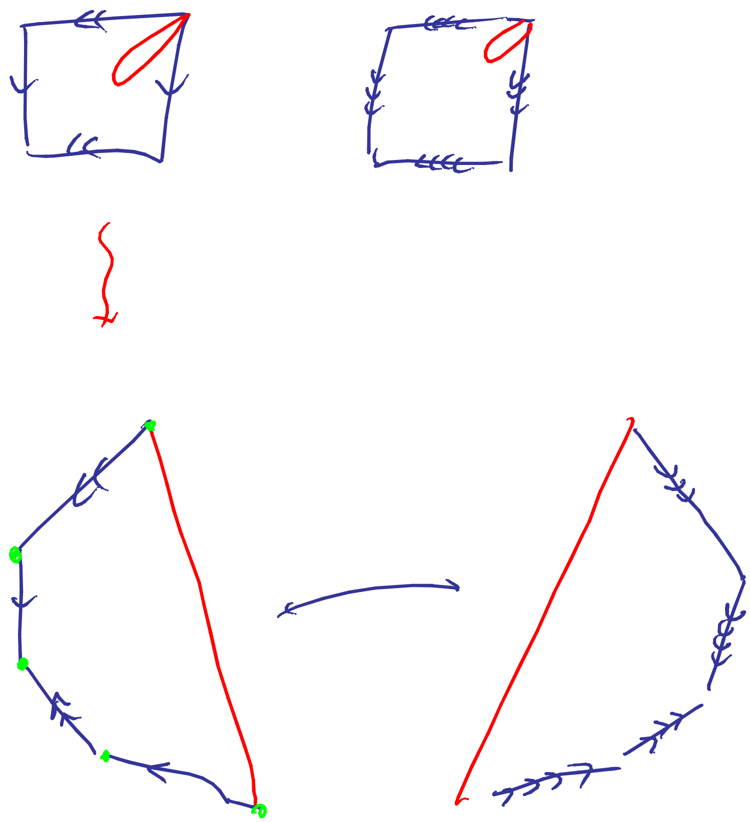
$\Sigma_g \# T^2$

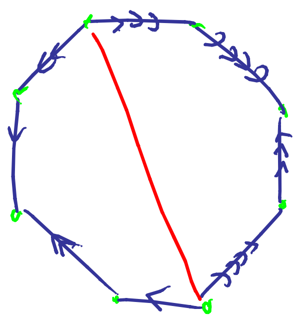


$=$ 

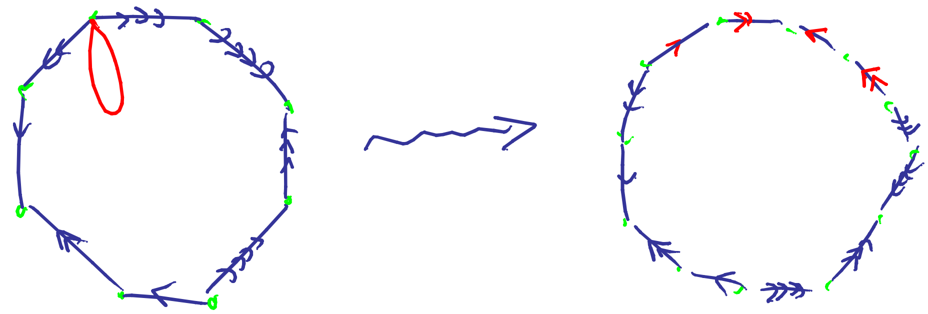
$g+1$

Polygonally





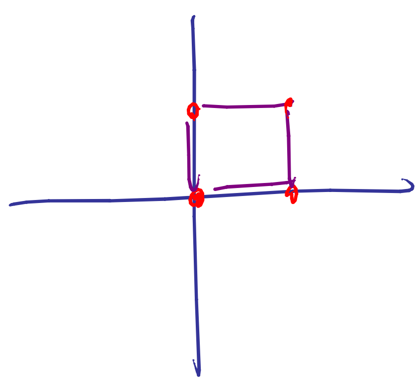
note all vertices
are identified



Σ'_3

all vertices are identified

Key property $\mathbb{R}^2$



- edges are geodesics
- angle around vertex adds up to 2π

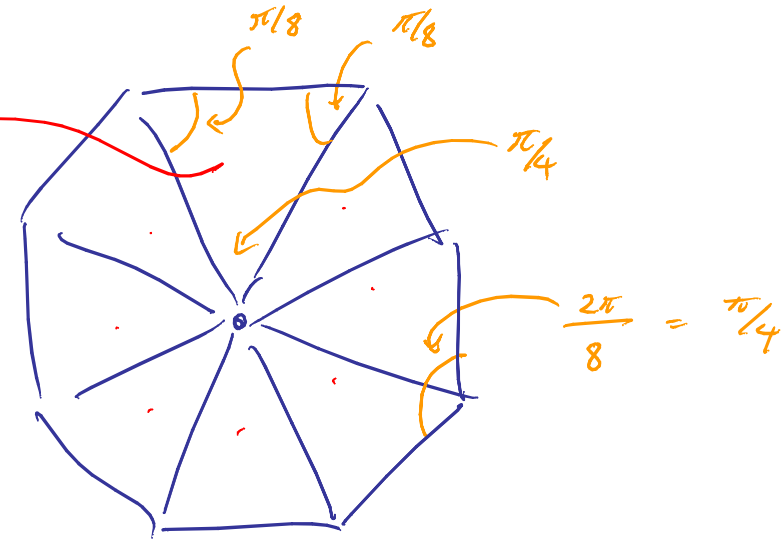
(only $\circ$ vertex)

$2\pi = \pi/2 \cdot 4$

$\int K\,dA$

$\parallel$

$-\pi/2$



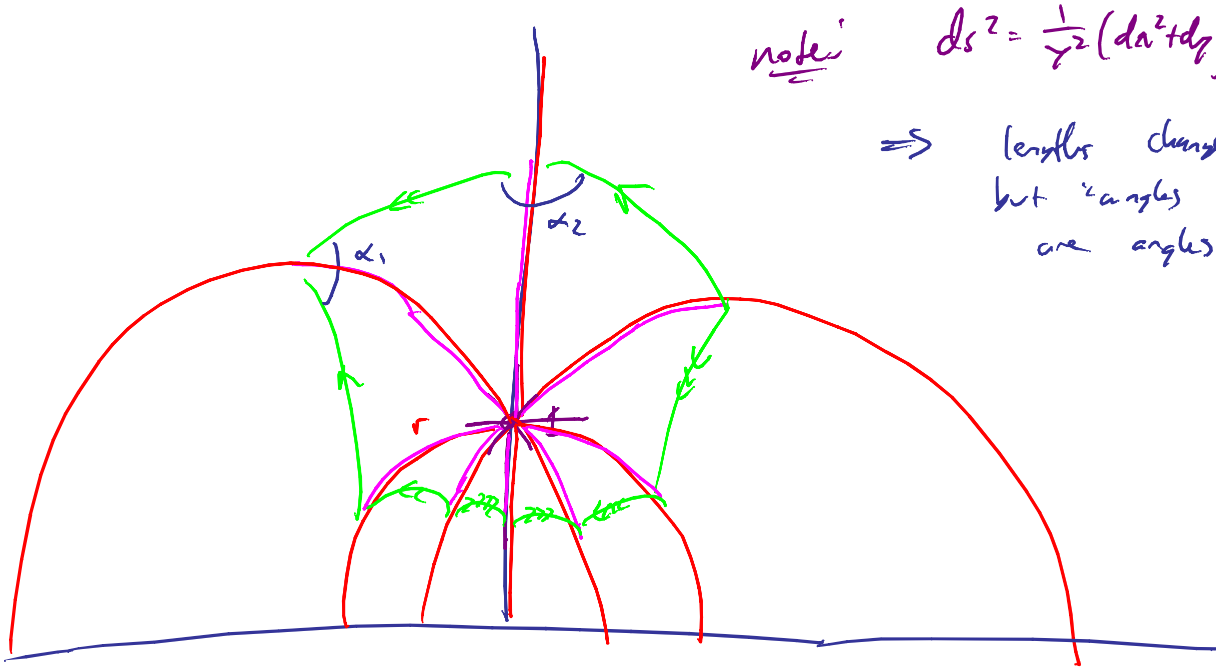
$$\int_{\Sigma_2} K\,dA = -\frac{8\pi}{2} = -4\pi = 2\pi(\chi(\Sigma_2))$$

$$\frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{4} = \frac{\pi}{2} < \pi$$

"Thin triangles"

note: $ds^2 = \frac{1}{y^2}(dx^2+dy^2)$

$\Rightarrow$ lengths change but "angles are angles"



"Rotational invariance"

$$z \longmapsto \frac{az+b}{cz+d}$$

$z = x+iy$

"three point property"