3 - Curves

Note Title 9/17/2009

Def: A regular parametrized curve is a mapping

$$c : I \longrightarrow \mathbb{R}^n$$
$$I \subset \mathbb{R}$$

tangent vector to t_0

such that $\dot{c} = \frac{dc}{dt} \neq 0 \quad \forall t.$

Note: if I is an open interval, this is same as a C^1 immersion

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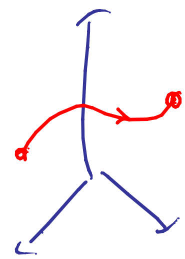
Length $I = [a, b]$

$$\text{Length}(c) = \int_a^b \left\| \frac{dc}{dt} \right\| dt$$

<u>lemma</u>: Length is independent of parametrization

$$\begin{array}{lll} t_2 & [a_1,b_1] & \xrightarrow{c_1} \\ & \Big\downarrow f & \\ t_2(t_1) & [a_2,b_2] & \xrightarrow{c_2} \end{array}$$



$$c_1(t_1) = c_2(t_2(t_1))$$

$$\int_{a_2}^{b_2} \left\| \frac{dc_2}{dt_2} \right\| dt_2 = \int_{a_1}^{b_1} \left\| \frac{dc_2}{dt_2} \right\| \frac{dt_2}{dt_1}\, dt_1$$

$$= \int_{a_1}^{b_1} \left\| \frac{dc_2}{dt_2} \right\| \left| \frac{dt_2}{dt_1} \right| dt_1$$

$$= \int_{a_1}^{b_1} \left\| \frac{dc_2}{dt_2} \frac{dt_2}{dt_1} \right\| dt_1$$

$$= \int_{a_1}^{b_1} \left\| \frac{dc_1}{dt_1} \right\| dt_1$$

Parametrization by arclength

Def $c(s) : [a,b] \longrightarrow \mathbb{R}^n$ regular parametrized curve

is parameterized by arclength

If $\left\| \frac{dc}{ds} \right\| = 1 \quad \forall s$

$$\left(\Rightarrow \quad s_0 \in a,b \qquad \int_a^{s_0} 1\, ds = s_0 - a \right.$$

$$\left. \mathrm{Length}\left(c\big|_{[a,s_0]} \right) = s_0 - a \right)$$

$$\mathrm{Length}(c) = b-a$$

Unique: up to shift $s \mapsto s+a$

The is unique $c : [0,L] \longrightarrow \mathbb{R}^n$

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lemma every regular curve admits parametrization by arclength. $c(t) : [a,b] \longrightarrow \mathbb{R}^n$

want $\bar{c}(s) : [0,L] \to \mathbb{R}^n$

Define $s(t) = \int_a^t \|\dot{c}(t_0)\| dt_0$

Q: how should I define $\bar{c}(s)$?

write $t(s)$

$$\bar{c}(s) = c(t(s))$$

Need to invert! ok inversing

is $\bar{c}(s)$ C' ?

$$\frac{ds}{dt} = \|\dot{c}(t)\|$$

$$\frac{d\bar{c}}{ds} = \frac{d}{ds} c(t(s))$$

$$= \dot{c}(t(s)) \frac{dt}{ds}$$

$$= \dot{c}(t(s)) \frac{1}{\left(\frac{ds}{dt}\right)}$$

$$= \frac{\dot{c}(t(s))}{\|\dot{c}(t(s))\|}$$ norm 1 ✓

$c(t)$, $c(s)$, $\dot{c} = \frac{dc}{dt}$, $c' = \frac{dc}{ds}$

Framed Curve:

Def Let $c : I \to \mathbb{R}^n$ be a regular parametrized curve.

An n-framing of c is a collection of vectors

$$(e_1(t), \ldots, e_n(t)) \qquad e_i \in \mathbb{R}^n$$

which depend continuously on t

s.t. $\forall\, t$, $(e_1(t), \ldots, e_n(t))$ is orthonormal.