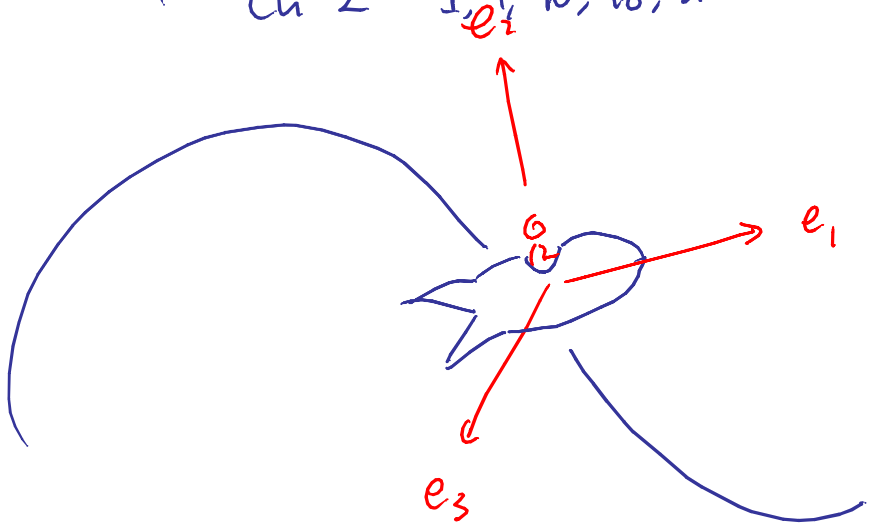
4 - Frenet Curves - Curvature - torsion

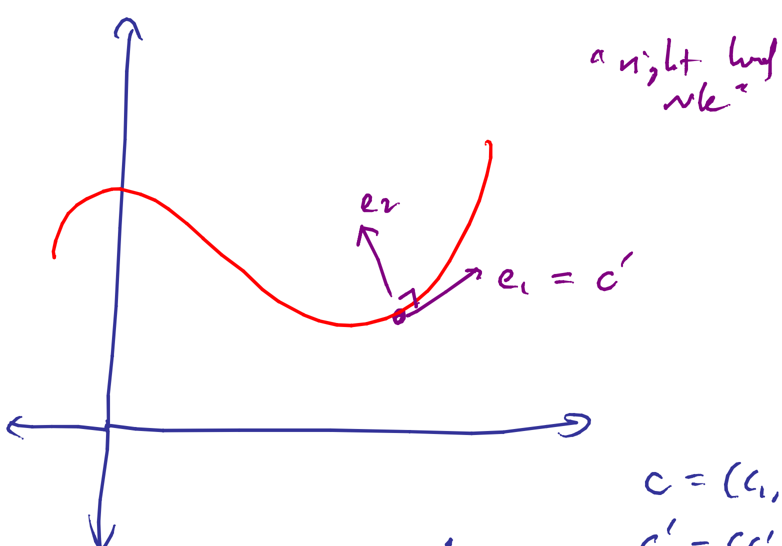
Note Title 9/17/2009

Pset 2: Ch 2 1, 4, 10, 18, 20



"Search for most natural frame"

$\underline{\underline{n=2}}$



$c = (c_1, c_2)$

$c' = (c_1', c_2')$

$e_1 = c'$

"Frenet 2-frame" $\quad e_2 = (-c_2', c_1')$

Suppose c is C^2

Q: what can you tell me about c''?

$$c''(s) = K(s)\, e_2(s)$$

why?

$$\|c'\| = 1$$

$$\Rightarrow \langle c', c' \rangle = 1$$

differentiate

$$\underbrace{\langle c'', c' \rangle + \langle c', c'' \rangle}_{= 2\langle c', c'' \rangle} = 0$$

$$\Rightarrow c'' \perp e_1$$

$$\Rightarrow c'' \text{ is a multiple of } e_2$$

Def $K(s)$ is the curvature of c at s

Recall: $(V, \langle -, - \rangle)$ e_i orthonormal basis

$$v = a_1 e_1 + \cdots + a_n e_n$$

$$a_i = \langle v, e_i \rangle$$

$$e_2' = \langle e_2', e_1 \rangle e_1 + \langle e_2', e_2 \rangle e_2$$

$$e_1' = \kappa e_2$$

$$\langle e_2, e_1 \rangle = 0 \implies \langle e_2', e_1 \rangle + \langle e_2, e_1' \rangle = 0$$

$$\implies \langle e_2', e_1 \rangle = -\langle e_2, \kappa e_2 \rangle = -\kappa$$

So $\boxed{\begin{aligned} e_1' &= \kappa e_2 \\ e_2' &= \kappa e_1 \end{aligned}}$

$$\boxed{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}' = \begin{bmatrix} & \kappa \\ -\kappa & \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}$$

Frenet equations

Application!

Thm Let c be a regular curve.

$\kappa(s)$ constant (nonzero) $\iff$ c is part of a circle of radius $\frac{1}{|\kappa|}$

$\kappa \equiv 0 \iff c$ is a line

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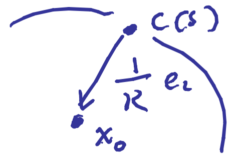
pf) $\kappa \equiv 0$ $\quad e_2' = 0 \implies e_2$ constant

$\implies e_1$ constant $\implies$ line

κ constant $\neq 0$.

<u>Claim:</u>



$$\left(c(s) + \frac{1}{\kappa} e_2\right)'$$

constant call it x_0

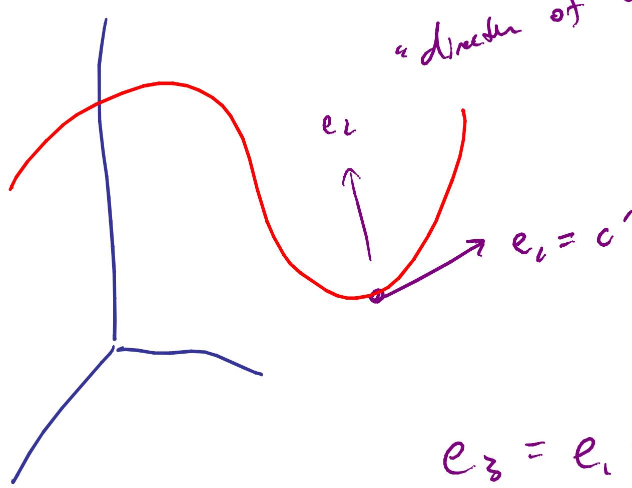
$$= e_1 + \frac{1}{\kappa} e_2' = e_1 - e_1 = 0$$

$c(s) = x_0 - \frac{1}{\kappa} e_2$

<u>$n=3$</u>

<u>Canonical</u> <u>framing</u> ?

" direction of c'' "



$e_3 = e_1 \times e_2$

Need $c'' \neq 0$
exist

<u>If so</u> define

$$\begin{cases} e_1 = c' & \text{tangent} \\ e_2 = \dfrac{c''}{\|c''\|} & \text{principal normal} \\ e_3 = e_1 \times e_2 & \text{binormal} \end{cases}$$

More generally

Def: c is a regular curve of class C^n

c is a Frenet curve if

$(c', c'', \ldots, c^{(n-1)})$ linearly independent

Get unique Frenet n-frame

- $(e_1, \ldots, e_n)$ orthonormal, positively oriented
- $\text{Span}(e_1, \ldots, e_k) = \text{Span}(c', \ldots, c^{(k)})$ $\quad 1 \leq k \leq n-1$
- $\langle c^{(k)}, e_k \rangle > 0$ $\quad 1 \leq k \leq n-1$

e_n = "generalized cross product"
$e_1 \times e_2 \times \cdots \times e_{n-1}$

determined by matrix: $\det \left[e_1 \middle| e_2 \middle| \cdots \middle| e_n \right] > 0$

$$e_1 = c'$$

$$e_2 = \frac{c''}{\|c''\|}$$

$$e_3 = \frac{\left(c''' - \langle c''', e_1\rangle e_1 - \langle e''', e_2\rangle e_2\right)}{\| - \|}$$

etc.-----

Space Curves:

Recall

c is Frenet if

$c' \neq 0$ $\quad\quad \langle c'', c'\rangle = 0$

$c'' \neq 0$

$$e_1 = c'$$

$$e_2 = \frac{c''}{\|c''\|}$$

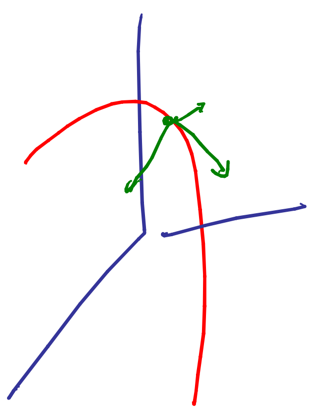
$$e_3 = e_1 \times e_2$$

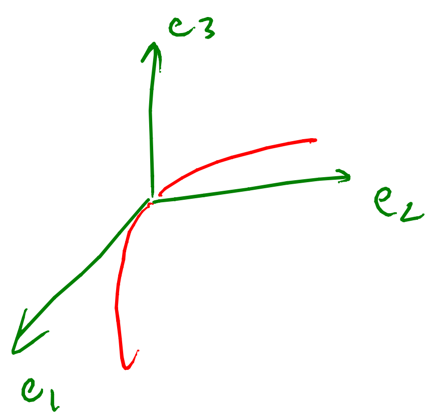
Frenet equations:

$$e_1' = c'' = \kappa e_2 \quad \Longrightarrow \quad \kappa = \|c''\|$$

curvature
always positive

Rmk: no longer talk about neg curvature
– depends on point of view
i.e. Frenet 3-frame





$$e_2' = \langle e_2', e_1\rangle e_1 + \underbrace{\langle e_2', e_2\rangle}_{0} e_2 + \langle e_2', e_3\rangle e_3$$

Since $\langle e_2, e_2\rangle = 1$

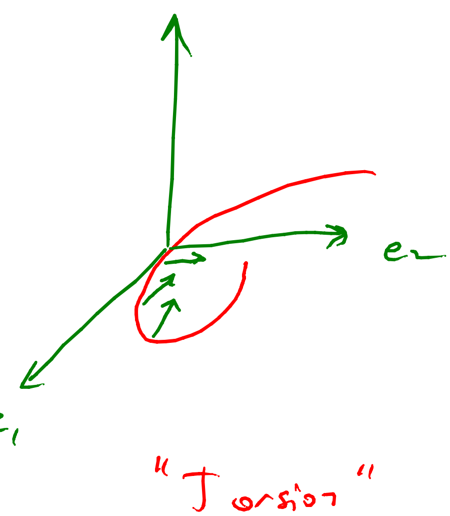
$$\xRightarrow{\frac{d}{ds}} \quad 2\langle e_2', e_2\rangle = 0$$

Now $\langle e_2, e_1 \rangle = 0$

$$\Rightarrow \langle e_2', e_1 \rangle + \langle e_2, e_1' \rangle = 0$$

$$\Rightarrow \langle e_2', e_1 \rangle = -\langle e_2, e_1' \rangle = -\langle e_2, \kappa e_2 \rangle$$
$$= -\kappa$$

$$\langle e_2', e_3 \rangle \overset{?}{=} \tau$$



sign = which way we lift

$$e_3' = \langle e_3', e_1 \rangle e_1 + \langle e_3', e_2 \rangle e_2 + \cancel{\langle e_3', e_3 \rangle e_3}$$

$\langle e_3, e_1 \rangle = 0$

$$\Rightarrow \langle e_3', e_1 \rangle = \langle e_3, e_1' \rangle = \langle e_3, \kappa e_2 \rangle$$
$$= 0$$

$\langle e_3, e_2 \rangle = 0$

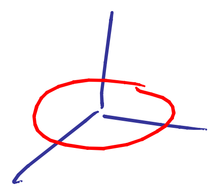
$$\Rightarrow \langle e_3', e_2 \rangle + \langle e_3, e_2' \rangle = 0$$
$$\Rightarrow \langle e_3', e_2 \rangle = -\tau$$

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

what can you tell me
about this matrix?

"Anti-symmetric"

example



$$c = (r\cos(\theta), r\sin(\theta), 0)$$

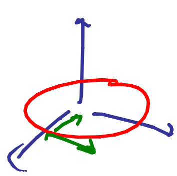
$$c(s) = \left(r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right), 0\right)$$

$$\Longrightarrow \quad c'(s) = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right), 0\right)$$

$$\Longrightarrow \quad c''(s) = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right), \frac{-1}{r}\sin\left(\frac{s}{r}\right), 0\right)$$

$$\kappa = \|c''\| = \frac{1}{r}$$

$$e_3 = (0,0,1)$$



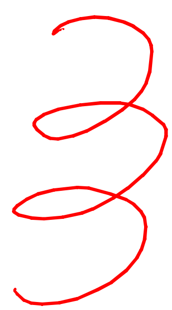
$$e_2 = \frac{c''}{\|c''\|} = \left(-\cos\left(\frac{s}{r}\right), -\sin\left(\frac{s}{r}\right), 0\right)$$

$$e_2' = \left(\frac{1}{r}\sin\left(\frac{s}{r}\right), -\frac{1}{r}\cos\left(\frac{s}{r}\right), 0\right)$$

$$= -\frac{1}{r}\, e_1$$

$$\tau = \langle e_2', e_3 \rangle = 0$$

Helix: $\quad c = (r\cos(\theta), r\sin(\theta), a\theta)$



speed $\quad \|\dot{c}\| = \sqrt{r^2 + a^2} \qquad$ Constant

$$c(s) = \left(r\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right)$$

$$c'(s) =$$

$$\left(-\frac{r}{\sqrt{r^2+a^2}} \sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \ \frac{r}{\sqrt{r^2+a^2}} \cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \ \frac{a}{\sqrt{r^2+a^2}} \right)$$

$$c''(s) =$$

$$\left(-\frac{r}{r^2+a^2} \cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \ \frac{-r}{r^2+a^2} \sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \ 0 \right)$$

So $\qquad \kappa = \|c''\| = \dfrac{r}{r^2+a^2}$

$$e_3 = e_1 \times e_2 = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{r}{\sqrt{r^2+a^2}} \sin & \frac{r}{\sqrt{r^2+a^2}} \cos & \frac{a}{\sqrt{r^2+a^2}} \\ -\cos & -\sin & 0 \end{pmatrix}$$

$$= \frac{a}{\sqrt{r^2+a^2}} \sin\left(\frac{s}{\sqrt{r^2+a^2}}\right) \hat{i} - \frac{a}{\sqrt{r^2+a^2}} \cos\left(\frac{s}{\sqrt{r^2+a^2}}\right) \hat{j}$$

$$+ \frac{r}{\sqrt{r^2+a^2}} \hat{k}$$

$$e_2' = \left(\frac{1}{\sqrt{r^2+a^2}} \sin , \frac{-1}{\sqrt{r^2+a^2}} \cos , 0 \right)$$

$$\langle e_2', e_3 \rangle = \frac{a}{r^2+a^2}$$

$$\alpha = \frac{1}{\sqrt{r^2+a^2}}$$

$$\alpha^2 a^2 = \frac{a^2}{r^2+a^2}$$

$$\kappa = \frac{r}{r^2+a^2}$$

$$\tau = \frac{a}{r^2+a^2}$$