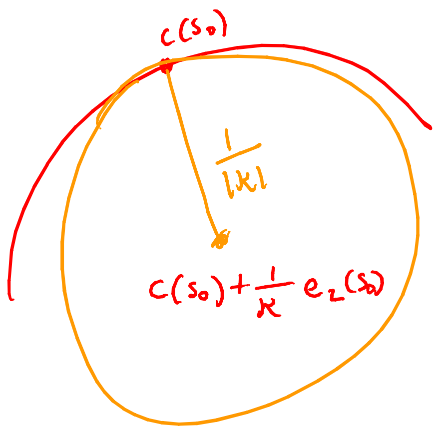
5- constraints on curvature - torsion

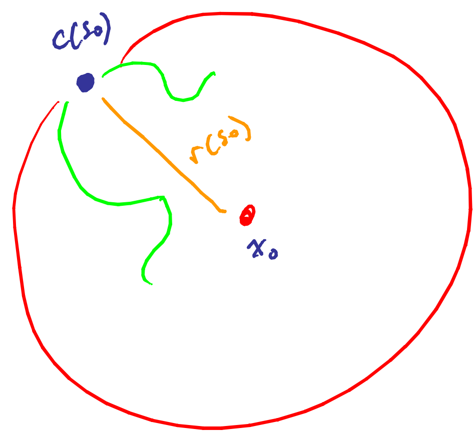
Note Title 9/22/2009

In $\mathbb{R}^2$ we had



κ constant $\Leftrightarrow$ c lies on circle

Thm (Osculating sphere)



$\tau(s_0) \neq 0$

$\Rightarrow$ there is a unique sphere such that the curve has a point of contact of third order

i.e. $r(s) = \|x_0 - c\|$

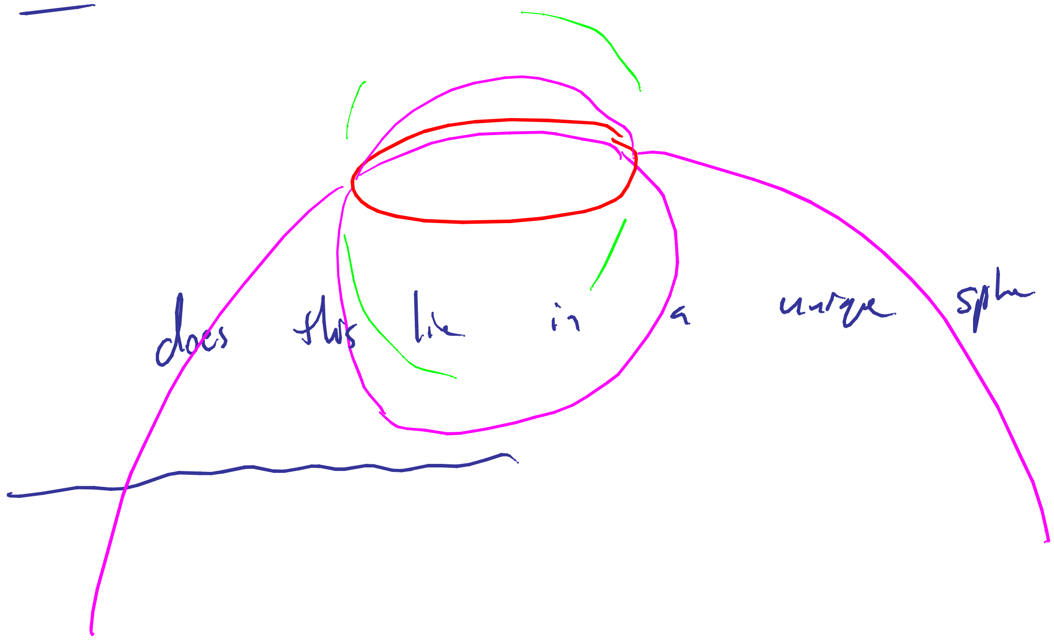
$$r'(s_0) = r''(s_0) = r'''(s_0) = 0$$

The unique sphere is characterized by

$$x_0 = c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\tau(s_0)\kappa^2(s_0)} e_3(s_0)$$

$$r(s_0) = \sqrt{\frac{1}{\kappa^2} - \frac{(\kappa')^2}{\tau^2\kappa^4}}$$

Q: consider circle



does this lie in a unique sphere?

(pf) We will instead consider

Write $x_0 = c(s_0) + \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0)$

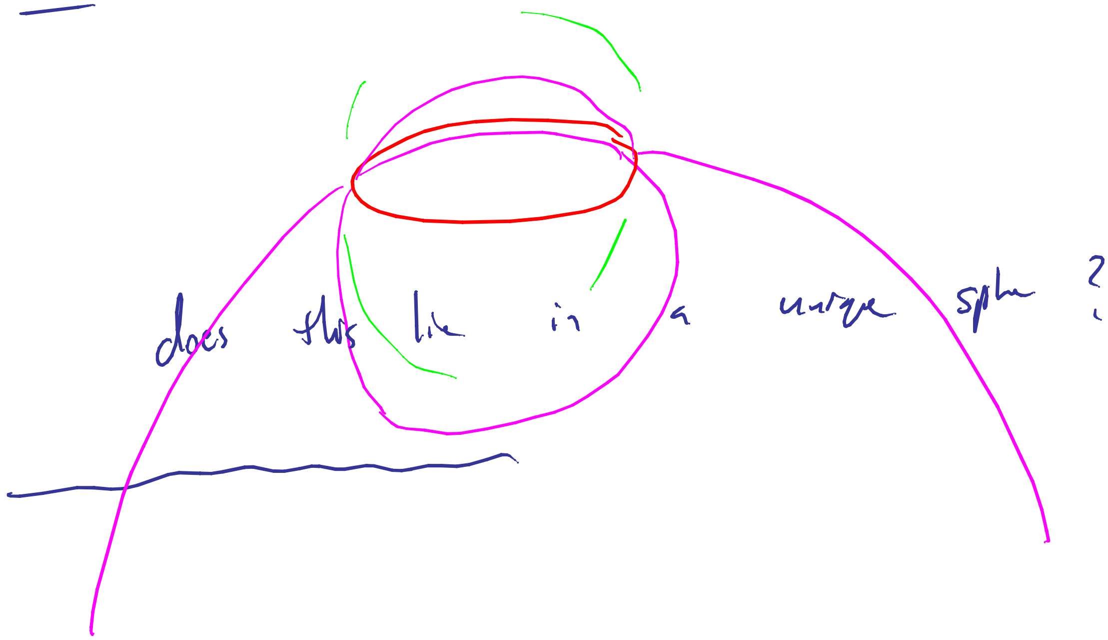
<u>Suffices to show</u> $f(s) = r(s)^2$

$$f'(s_0),\ f''(s_0),\ f'''(s_0) = 0$$

$$f(s) = \langle x_0 - c(s),\ x_0 - c(s) \rangle$$

$$f'(s) = 2\langle x_0 - c(s),\ -c'(s) \rangle$$

Q: consider circle



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$$f'(s) = 2\langle x_0 - c(s), -c'(s) \rangle$$

$$f''(s) = 2\langle -c'(s), -c'(s)\rangle + 2\langle x_0 - c(s), -c''(s)\rangle$$

$$f'''(s) = 4\langle -\cancel{c'}(s), -c''(s)\rangle + 2\langle -c'(\cancel{s}), -c''(s)\rangle$$
$$+ 2\langle x_0 - c(s), -c'''(s)\rangle$$

$$x_0 - c(s_0) = \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0)$$

$$f'(s_0) = 0 \iff 2\langle x_0 - c(s_0), -e_1\rangle = 0$$

$$\iff \alpha = 0$$

$$f''(s_0) = 0 \iff 2\overset{1}{\overset{\shortparallel}{\langle e_1, e_1\rangle}} + \overset{-2\beta\kappa}{\overset{\shortparallel}{2\langle x_0 - c(s_0), -\kappa e_2\rangle}}$$
$$\overset{\shortparallel}{0}$$

$$\iff \beta\kappa = 1$$

$$\iff \beta = \frac{1}{\kappa}$$

$$f'''(s_0) = 0 \iff \langle x_0 - c(s_0), -c'''(s_0)\rangle = 0$$

$$c'' = \kappa e_2$$

$$c''' = \kappa' e_2 + \kappa e_2'$$

$$= \kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3$$

$$\Longleftrightarrow \quad \frac{\kappa'}{\kappa} + \kappa\tau\gamma = 0$$

$$\Longrightarrow \quad \gamma = -\frac{\kappa'}{\kappa^2\tau}$$

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c frenet, class C^4, $\tau \neq 0$

Thm c lies on a sphere iff

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'$$

(pf) $\quad x_0(s) = c(s) + \frac{1}{\kappa} e_2 - \frac{\kappa'}{\tau\kappa^2} e_3$

c lies on sphere iff $x_0(s)$ is constant

$$x_0' = e_1 + \left(-e_1 + \frac{\tau}{\kappa} e_3\right) - \frac{\kappa'}{\kappa^2} e_2$$

$$- \left(\frac{\kappa'}{\tau\kappa^2}\right)' e_3 + \frac{\kappa'}{\kappa^2} e_2$$

$$= \left(\frac{\tau}{\kappa} - \frac{\kappa'}{\tau\kappa^2}\right)' e_3 = 0$$

$$\iff \frac{\tau}{\kappa} = \frac{\kappa'}{\tau\kappa^2}$$

Thm c C^3 curve in S^2 (unit sphere)

J: $\det \begin{bmatrix} c \\ c' \\ c'' \end{bmatrix}$

$\Rightarrow$ c is Frenet

and $\kappa = \sqrt{1+J^2}$

$\tau = \dfrac{J'}{(1+J^2)}$

$J \equiv 0 \iff c$ is a great circle

$J \equiv$ const $\iff c$ is a circle

<u>Note:</u> $\langle c, c\rangle = 1 \Rightarrow \langle c, c'\rangle = 0$

So $c, c', c\times c'$ is an orthonormal 3-frame

$$c'' = \langle c'', c\rangle c + \cancel{\langle c'', c'\rangle} c' + \underbrace{\langle c'', c\times c'\rangle}_{J} c\times c'$$

$$\boxed{\det\begin{bmatrix} c'' \\ c \\ c' \end{bmatrix}}$$

$$\langle c', c\rangle = 0$$

$$\Rightarrow \langle c'', c\rangle + \overbrace{\langle c', c'\rangle}^{1} = 0$$

$$\Rightarrow \langle c'', c\rangle = -1$$

So $c'' = -c + J\, c\times c'$

$$K = \|c''\| = \sqrt{1+J^2}$$

$$e_2 = \frac{1}{K} c'' \qquad e_3 = c'\times e_2$$

$$\langle c'', c\rangle = -1 \Rightarrow \langle c''', c\rangle = -\overset{0}{\cancel{\langle c'', c'\rangle}}$$

$$\tau = -\langle e_3', e_2 \rangle = \left\langle \left(\frac{1}{\kappa} c' \times c''\right)', \frac{1}{\kappa} c'' \right\rangle$$

$$= -\frac{1}{\kappa^2} \langle c' \times c''', c'' \rangle$$

$$+ \frac{\kappa'}{\kappa^3} \cancel{\langle c' \times c'', c'' \rangle}$$

$$= -\frac{1}{\kappa^2} \langle c' \times c''', -c + J c \times c' \rangle$$

$$= -\frac{1}{\kappa^2} \langle c' \times c''', -c \rangle$$

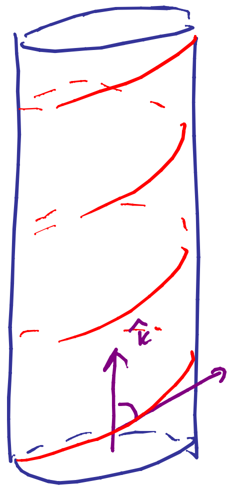
$$= \frac{J'}{\kappa^2}$$

$$J' = \langle c' \times c'', c \rangle' = \langle c' \times c''', c \rangle$$

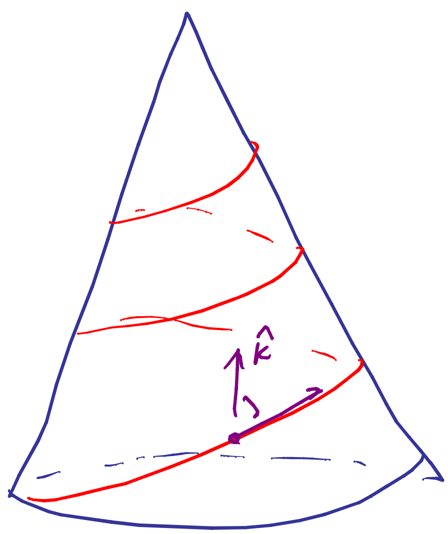
Rmk: Geodesic $\Longleftrightarrow$ great circle

Geodesic curvature is 1

Slope lines:



constant angle



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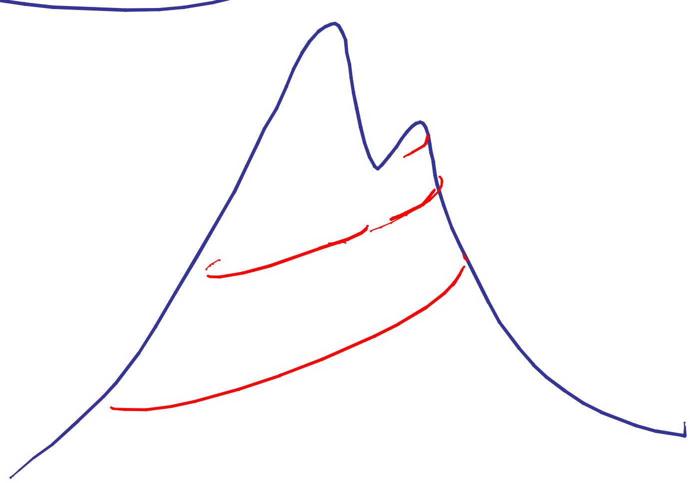

Def: c = curve in $\mathbb{R}^3$ is a **slope line** if there is a vector $v \in \mathbb{R}^3$, $v \neq 0$

s.t. $\langle c', v \rangle$ is constant

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c = Frenet curve

Thm: T.F.A.E.

(i) c is a slope line

(ii) $\exists\ v \neq 0$ w/ $\langle e_2, v \rangle = 0$

(iii) $\exists\ v \neq 0$ w/ $\langle e_3, v \rangle$ constant

(iv) $\frac{\tau}{\kappa}$ constant

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e.g. Helix what is τ ?

$$c(s) = \left(r \cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), r\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{as}{\sqrt{r^2+a^2}}\right)$$

$$e_1 = \left(-\frac{r}{\sqrt{r^2+a^2}} \sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{r}{\sqrt{r^2+a^2}} \cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), \frac{a}{\sqrt{r^2+a^2}}\right)$$

$$e_2 = \left(-\cos\left(\frac{s}{\sqrt{r^2+a^2}}\right), -\sin\left(\frac{s}{\sqrt{r^2+a^2}}\right), 0\right)$$

earlier

$$e_3 = \left(\frac{a}{\sqrt{r^2+a^2}} \sin(-),\ -\frac{a}{\sqrt{r^2+a^2}} \cos,\ \frac{r}{\sqrt{r^2+a^2}} \right)$$

$$\kappa = \frac{r}{r^2+a^2}$$

$$\tau = \frac{a}{r^2+a^2}$$

$$\tau e_1 + \kappa e_3 = \sqrt{r^2+a^2}\ \hat{k} = D$$

Assume $\tau \neq 0$ ($\tau = 0 \Rightarrow$ plane curve)

(pf) (i) $\Leftrightarrow$ (ii)

$$\langle e_1, v\rangle' = \langle \kappa e_2, v\rangle \qquad \kappa \neq 0$$

(ii) $\Leftrightarrow$ (iii)

$$\langle e_3, v\rangle' = \langle -\tau e_2, v\rangle$$

(i) $-$ (iii) $\Rightarrow$ $v = \alpha e_1 + \beta e_3$ $\quad \alpha, \beta$ constant

$$0 = \alpha e_1' + \beta e_3'$$

$$= \alpha \kappa e_2 - \beta \tau e_2$$

$$\Rightarrow \alpha \kappa = \beta \tau \Rightarrow \frac{\tau}{\kappa} = \frac{\alpha}{\beta}$$

constant (iv)

(iv) $\Rightarrow$ (i)$-$(iii)

Conversely: If $\frac{\tau}{\kappa}$ constant

Set $\alpha = \frac{\tau}{\kappa}$

$\beta = 1$

$$V(s) = \frac{\tau}{\kappa} e_1(s) + e_3(s) \qquad \text{independent of } s \text{ ?}$$

$$V' = \frac{\tau}{\kappa} \kappa e_2 - \tau e_3 = 0 \qquad \checkmark$$

$\square$

Note $$D = \tau e_1 + \kappa e_3$$

$$= \kappa V$$