6 - General Frenet equations

Note Title 9/24/2009

Thm: c is a Frenet curve in $\mathbb{R}^n$

$$\begin{bmatrix} e_1' \\ \vdots \\ e_n' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & \cdots & & 0 \\ -\kappa_1 & 0 & \kappa_2 & & & \vdots \\ 0 & -\kappa_2 & 0 & \kappa_3 & & \vdots \\ \vdots & & -\kappa_3 & \ddots & & \\ & & & & 0 & \kappa_{n-1} \\ 0 & \cdots & & & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$\kappa_1, \ldots, \kappa_{n-2} > 0$$

(pf) $e_i' = \langle e_i', e_1 \rangle e_1 + \cdots + \langle e_i', e_n \rangle e_n$

i.e. $$\begin{bmatrix} e_1' \\ \vdots \\ e_n' \end{bmatrix} = \underbrace{\Big[\langle e_i', e_j \rangle \Big]_{i,j}}_{= A} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$\langle e_i, e_j \rangle = \underset{0,1}{\text{const}}$

$\Rightarrow \langle e_i', e_j \rangle + \langle e_i, e_j' \rangle = 0$

$\Rightarrow \langle e_i', e_j \rangle = -\langle e_j', e_i \rangle$

$\Rightarrow$ A skew symm.

$\Rightarrow$ just need to determine

$\langle e_i', e_j \rangle$ for $j > i$

Define $\kappa_i = \langle e_i', e_{i+1} \rangle$

suppose $j > i+2$

$e_i \in \mathrm{Span}(c', \ldots, c^{(i)})$

$\Rightarrow e_i' \in \mathrm{span}(c'', \ldots, c^{(i+1)}) \subset \mathrm{Span}(c', \ldots, c^{(i+1)})$

$\|$

$\mathrm{Span}(e_i, \ldots, e_{i+1})$

$\Rightarrow \langle e_i', e_j \rangle = 0$ for $j > i+2$

$\square$

Note: c is contained in a

hyperplane $\Leftrightarrow$ $\kappa_{n-1} \equiv 0$

(n-1 dimensional sub-plane)

<u>Rmk</u>: κ_i invariant under rotation, translation.

<u>Thm</u> (Fund thm of local theory of curves)

$\kappa_1, \dots, \kappa_{n-1} : (a,b) \longrightarrow \mathbb{R}$ $\quad C^\infty$

$\kappa_1, \dots, \kappa_{n-2} > 0$

$s_0 \in (a,b)$, $\quad q_0 \in \mathbb{R}^n$, $\quad \bar{e}_1, \dots, \bar{e}_n$

$\exists!$ C^∞- Frenet curve parametrized by arclength $\quad c : (a,b) \longrightarrow \mathbb{R}^n$

(1) $c(s_0) = q_0$

(2) $e_i(s_0) = \bar{e}_i$

(3) κ_i = Frenet curvatures of c

(pf)

$$E(s_0) = \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$K(s) = \begin{bmatrix} & \kappa_1 & & & \\ -\kappa_1 & & \kappa_2 & & \\ & -\kappa_2 & \ddots & & \\ & & \ddots & & \kappa_{n-1} \\ & & & -\kappa_{n-1} & \end{bmatrix}$$

$$K \cdot E = \begin{bmatrix} \kappa_1 e_2 \\ -\kappa_1 e_1 + \kappa_2 e_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_1' \\ e_2' \\ \vdots \\ e_n' \end{bmatrix}$$

Recall: ODE

$$\begin{bmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{bmatrix} = A(t) \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$t \in (a,b)$

Always has a unique sol'n w/ initial condition $\begin{bmatrix} f_1(t_0) \\ \vdots \\ f_n(t_0) \end{bmatrix}$

$t_0 \in (a,b)$

$$\Longrightarrow \begin{bmatrix} f_{11}' & f_{12}' & \cdots & f_{1,n}' \\ \vdots & \vdots & & \vdots \\ f_{n,1}' & f_{n,2}' & \cdots & f_{n,n}' \end{bmatrix} = A(t) \begin{bmatrix} f_{11} & \cdots \\ \vdots & \\ f_{n,1} & \end{bmatrix}$$

has unique soln. given $[f_{ij}(t_0)]$

$$\Longrightarrow \text{given initial cndtn} \begin{bmatrix} \bar{e}_1 \\ \vdots \\ \bar{e}_n \end{bmatrix} = \bar{E}$$

$$\exists! \underbrace{\begin{bmatrix} e_1(s) \\ \vdots \\ e_n(s) \end{bmatrix}}_{E(s)}$$

s.t. 1) $E'(s) = K(s)\,E(s)$

2) $E(s_0) = \bar{E}$

Q: is E ortonormal ?

$$E E^t = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = I_n$$

$$\bar{E} \bar{E}^t = I_n \quad \text{(by asumption)}$$

$$(E E^t)' = E' E^t + E (E^t)'$$

$$E' E^t = K E E^t$$

$$E (E^t)' = E (E')^t = E (K E)^t$$

$$= E E^t K^t$$

$$= - E E^t K$$

Let $B(s) = E E^t$

$$B(s_0) = \bar{E} \bar{E}^t = I_n$$

$$B' = K B - B K$$

Try $I_n = B$

$$I_n' = 0 \quad \checkmark$$

$$K I_n - I_n K = K - K = 0$$

uniqueness $\Rightarrow$ $B = I_n$

$$\overset{\shortparallel}{EE^t}$$

So we have $e_i(s)$ orth. frame.

<u>Solve!</u>

$$c' = e_1$$

$$c(s) = q_0 + \int_{s_0}^{s} e_1(t)\,dt$$

<u>check</u> $\quad c'' = e_1' = \kappa_1 e_2 \qquad \kappa_1 > 0$

$\Rightarrow$ e_2 is part of

Frenet frame

$$c''' = e_1'' = (\kappa_1 e_2)' = \kappa_1' e_2 - \kappa_1 e_1 + \kappa_2 e_3$$

project away from e_1, e_2 $\quad e_3 \quad$ etc.....

Go backwards to see $c', c'', \dots, c^{(n-1)}$ (L.I.)

Example Every (Frenet) curve in $\mathbb{R}^3$ w/ constant κ and τ is a helix.