8 - Surfaces

(Only doing this because book does)

Def: $U \subseteq \mathbb{R}^2$ open

$f : U \to \mathbb{R}^3$ is a parametrized surface elmt

if it is an immersion

Q: is the image a submfld?

Def a (smooth) surface is a (C^∞) 2-submfld of $\mathbb{R}^3$

[But local parametrizations give parametrized surface elements.]

Def: (First fund. form - a.k.a Metric)

$M \hookrightarrow \mathbb{R}^3$ surface, $x \in M$

$I(-,-) : T_x M \times T_x M \to \mathbb{R}$ bilinear form

$I(v,w) = \langle v, w \rangle$ (v, w viewed as vectors in $\mathbb{R}^3$)

Given $f : U \longrightarrow M \subset \mathbb{R}^3$

local parametrization around x

$$Df\big|_u : T_u U \xrightarrow[\cong]{} T_{f(u)} M$$

get:

$$I(-,-) : T_u U \times T_u U \longrightarrow \mathbb{R}$$

$$(v, w) \longmapsto \left\langle Df\big|_u v \, , Df\big|_u w \right\rangle$$

Note: $T_u U = \mathbb{R}^2$

But: the inner product $I(-,-)$ on $T_u U$ is **NOT** the same as the standard "dot" product

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Explicit formulas

Recall If $\langle -, - \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

is an inner product

then $\exists$ matrix $A = [a_{ij}]$

So that

$$\langle x, y \rangle = [x_1 \cdots x_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= x^T A x$$

- $a_{ij} = \langle e_i, e_j \rangle$ $\quad e_1, \ldots, e_n$ = standard basis
- A is symmetric, and positive (i.e. has positive eigenvalues)

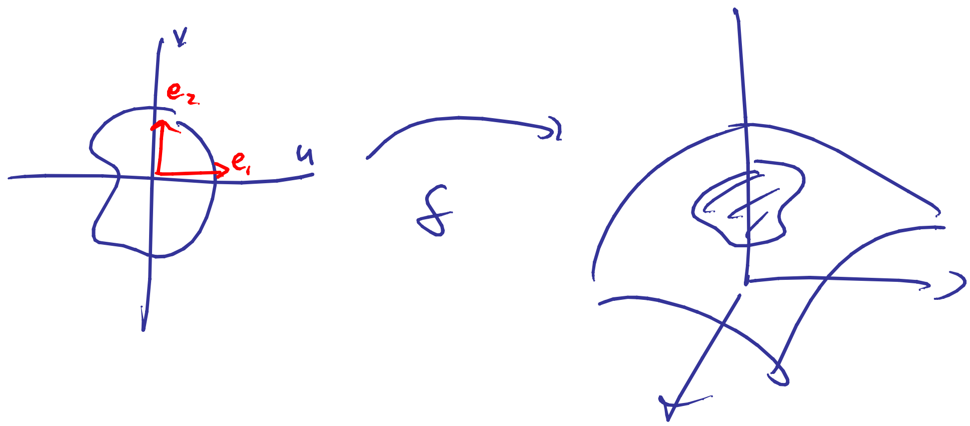
We say $\langle -, - \rangle \sim A$

Question

$- \cdot -$

dot product $\sim$ what matrix?

In explicit coordinates



$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{bmatrix}$$

$$Q: \quad Df\, e_1 = ? \qquad \frac{\partial f}{\partial u}$$

$$Df\, e_2 = \frac{\partial f}{\partial v}$$

$$I_f(-,-) \sim \left[\langle Df|_u e_i, Df|_u e_j \rangle \right]$$

$$= \begin{bmatrix} \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial u} \rangle & \langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \rangle \\ \langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial u} \rangle & \langle \frac{\partial f}{\partial v}, \frac{\partial f}{\partial v} \rangle \end{bmatrix} =: \left[g_{ij}(u,v) \right]$$

Not Identity

$\Longrightarrow$ different from "dot product"

Change of coordinates

$$\tilde{f} = f \circ \varphi$$

$$\begin{tikzcd}
(s,t) \quad \mathbb{R}^2 \supseteq V \arrow[d, "\varphi", "\approx"'] \arrow[rd, "\tilde{f}"] & \\
(u,v) \quad \mathbb{R}^2 \supseteq U \arrow[r, "f"'] & M
\end{tikzcd}$$

$$\varphi : \quad \begin{aligned} u &= u(s,t) \\ v &= v(s,t) \end{aligned}$$

Q: how is g_{ij} related to $\tilde{g}_{ij}$?

Lemma $$[\tilde{g}_{ij}] = (D\varphi)^T \cdot [g_{ij}] \cdot D\varphi$$

(pf) write all vectors as column vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$D(\tilde{f}) = D(f \circ \varphi) = Df \cdot D\varphi$$

$$\begin{tikzcd}
T_p V \arrow[d, "D\varphi"'] \arrow[dr, "D\tilde{f}"] & \\
T_{\varphi(p)} U \arrow[r, "Df"'] & T_{\tilde{f}(p)} M
\end{tikzcd}$$

take $v, w \in T_p V$

suffices to show $I_{\tilde{f}}(-,-) \sim (D\varphi)^T [g_{ij}] (D\varphi)$

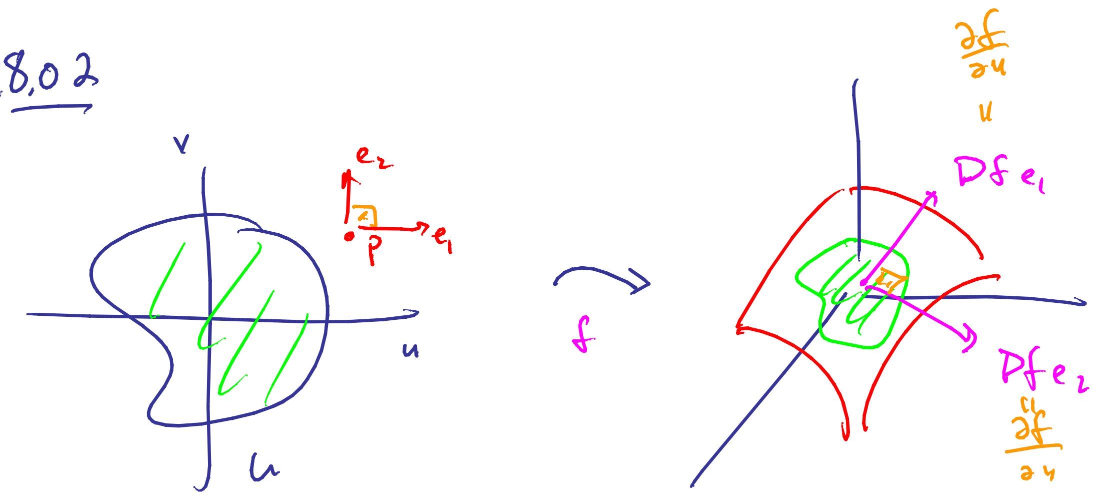
$$\begin{aligned}
I_{\tilde{f}}(v, w) &= \langle D\tilde{f}\, v,\ D\tilde{f}\, w \rangle \\
&= \langle Df(D\varphi\, v),\ Df(D\varphi\, w) \rangle \\
&= (D\varphi \cdot v)^T [g_{ij}] (D\varphi \cdot w) \\
&= v^T \underline{D\varphi^T [g_{ij}] D\varphi}\ w
\end{aligned}$$

$$I_{\tilde{f}}(-,-) \sim D\varphi^T [g_{ij}] D\varphi$$

$\square$

<u>Surface</u> <u>elements</u>

<u>18.02</u>



Surface integral

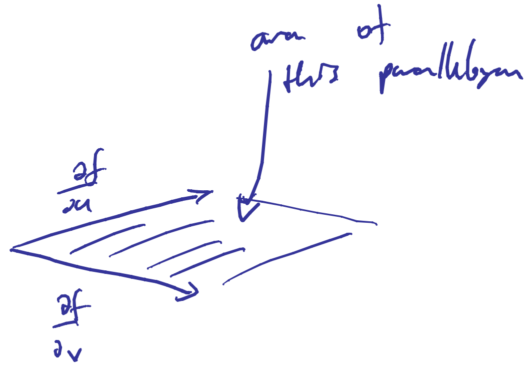
$h : M \to \mathbb{R}$

$$\int_M h \, dA$$

$$dA = \left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\| du\,dv$$

$$\left\langle \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \, , \, \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\rangle$$

$$\| \qquad \det(g_{ij})$$



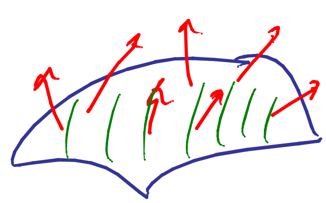
<u>Vector</u> <u>fields</u> [Defs work for k-submanifolds in $\mathbb{R}^n$]

<u>Def</u> Let $f : U \longrightarrow M \subseteq \mathbb{R}^3$ be a local parametrization, $U \subseteq \mathbb{R}^2$

element

A <u>vector</u> <u>field</u> <u>along</u> <u>f</u> is a map $X : U \to \mathbb{R}^3$

We think of $X_{(u,v)} \in T\mathbb{R}^3_{f(u,v)}$



X is <u>tangential</u> if $X(u,v) \in T_{f(u,v)}M$

" <u>normal</u> if $X(u,v) \perp T_{f(u,v)}M$

$M \subseteq \mathbb{R}^3$ is a surface

Def A (tangential) <u>vector field</u> on M is a mapping $X : M \longrightarrow \mathbb{R}^3$

s.t.

- $X(p) \in T_pM \subseteq T_p\mathbb{R}^3 \doteq \mathbb{R}^3$
- For every local parameterization

$$f : \underset{\underset{\mathbb{R}^2}{\cap}}{U} \longrightarrow M$$

the composition $U \longrightarrow M \longrightarrow \mathbb{R}^3$ is a vector field along f (i.e. C^∞)

<u>e.g.</u>

$$\begin{tikzcd}
\mathbb{R}^2 \supseteq U \arrow[d, "\varphi", "\approx"'] \arrow[dr, "f"] & \\
\mathbb{R}^2 \supseteq V \arrow[r, "\tilde{f}"'] & M
\end{tikzcd}$$

$X \circ f = X \circ \tilde{f} \circ \varphi$

$X \circ \tilde{f} = X \circ f \circ \varphi^{-1}$

$\Rightarrow X \circ f \ C^\infty \Leftrightarrow \underset{C^\infty}{X \circ \tilde{f}}$

So really just need to check for an atlas

(collection of local parametrizations which cover M)

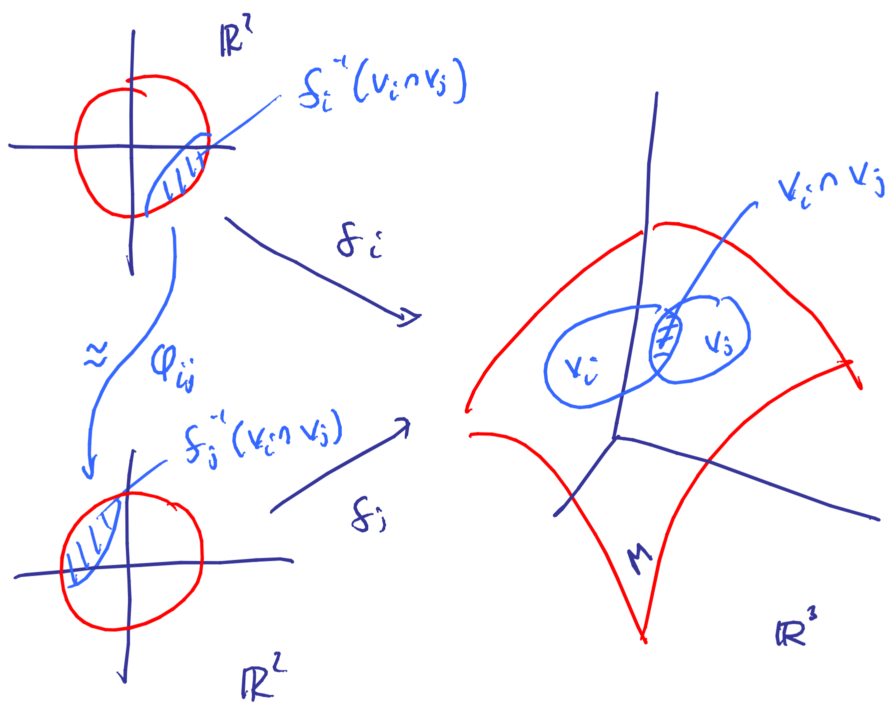
Atlas

$$f_i : U_i \longrightarrow M \subseteq \mathbb{R}^3 \qquad U_i \subseteq \mathbb{R}^2$$

local parametrizations

is an atlas if

$$M = \bigcup V_i \qquad V_i = f(U_i)$$



$\varphi_{ij} : f_i^{-1}(V_i \cap V_j) \to f_j^{-1}(V_i \cap V_j)$ is the Transition function

Def

$$\begin{array}{ccc} g : U & \xrightarrow{\cong} & V \\ \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^n \end{array} \quad \text{diffeo}$$

is <u>orientation</u> <u>preserving</u> if

$$\det J_x g > 0 \quad \text{for all } x \in \mathbb{R}^n$$

Def: M is <u>orientable</u> if

$\exists$ atlas $\{f_i : U_i \longrightarrow M\}$

whose transition functions φ_{ij} are all orientation preserving

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Special for <u>surface</u>

Lem:

$\underset{\text{surface}}{M \subseteq \mathbb{R}^3}$ is orientable if and only if M possesses a unit normal vector field.

(pf) $(\Longrightarrow)$ Take $X = \dfrac{\frac{\partial f_i}{\partial u} \times \frac{\partial f_i}{\partial v}}{\left\| \frac{\partial f_i}{\partial u} \times \frac{\partial f_i}{\partial v} \right\|}$

note this is independent of coordinate chart

$(\hat{u}, \hat{v})$ standard basis for $\mathbb{R}^2$

$$f_i = f_j \circ \varphi_{ij}$$

$$\Longrightarrow \quad \frac{\partial f_i}{\partial u} = Df_i\, \hat{u} = Df_j \cdot D\varphi_{ij}\, \hat{u}$$

$$\frac{\partial f_i}{\partial v} = Df_j \cdot D\varphi_{ij}\, \hat{v}$$

$$\frac{\partial f_i}{\partial u} \times \frac{\partial f_i}{\partial v} = \left[Df_j \cdot D\varphi_{ij}\, \hat{u} \right] \times \left[Df_j \cdot D\varphi_{ij}\, \hat{v} \right]$$

$$= \det \varphi_{ij} \left(Df_j\, \hat{u} \times Df_j\, \hat{v} \right)$$

$$= \det \varphi_{ij} \left(\frac{\partial f_j}{\partial u} \times \frac{\partial f_j}{\partial v} \right)$$

($\Leftarrow$) Pick atlas $\{f_i : U_i \longrightarrow M\}$

$$\frac{\dfrac{\partial f_i}{\partial u} \times \dfrac{\partial f_i}{\partial v}}{\left\| \text{———} \right\|} = \pm X$$

$+ \implies$ Take $\tilde{f}_i(u,v) = f_i(u,v)$

$- \implies$ take $\tilde{f}_i(u,v) = f_i(v,u)$

$\square$