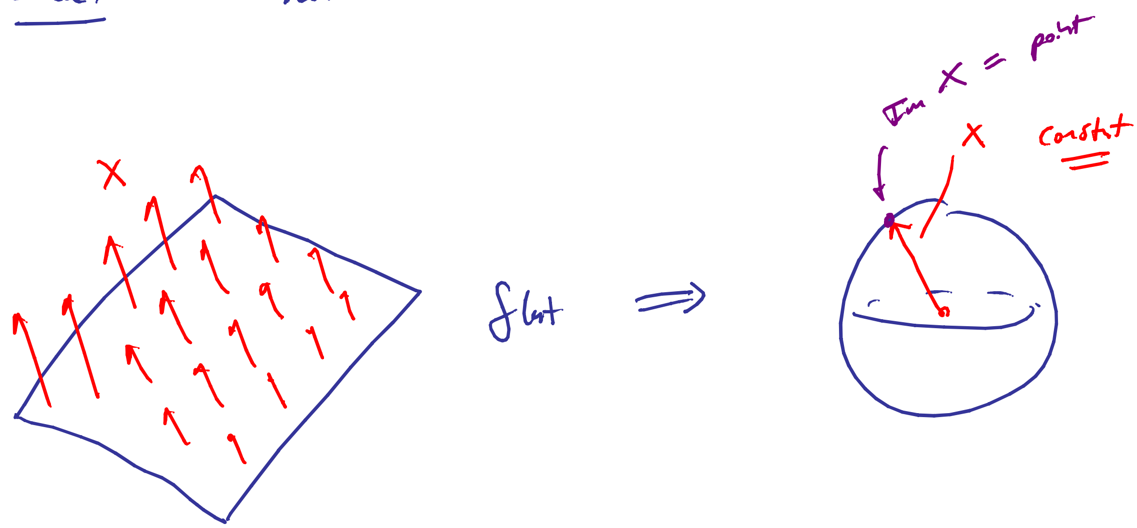
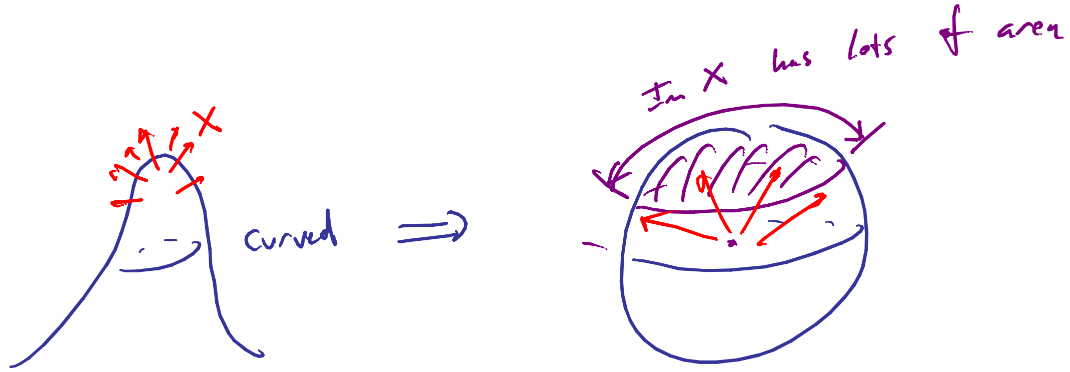
9 - Shape Operator

Note Title 10/6/2009

Idea: look at unit normals!



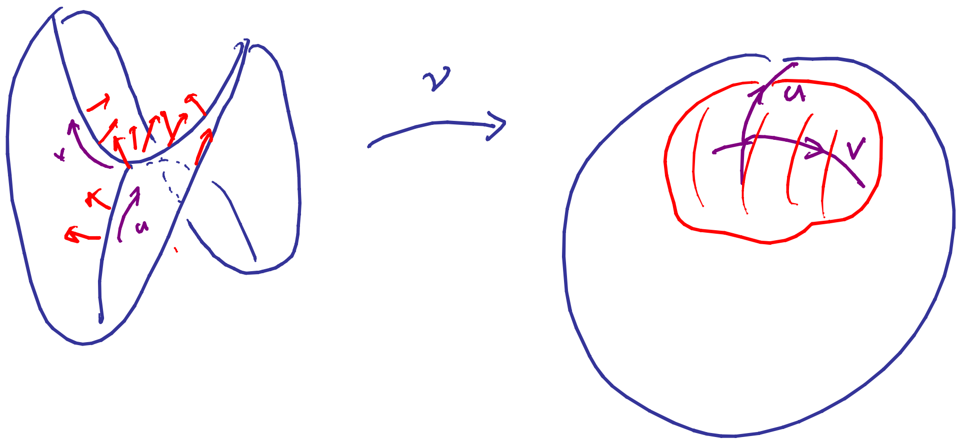


Gauss map

Def $f : U \longrightarrow M \subseteq \mathbb{R}^3$ local parametrization, $U \subseteq \mathbb{R}^2$

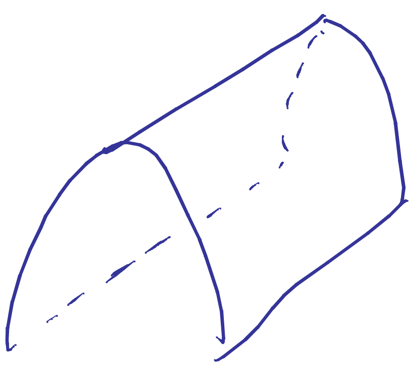
$$\nu : U \longrightarrow S^2$$

$$\nu(u,v) = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\left\| \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \right\|}$$



Note: ν orientation reversing!

"negative area" $\Rightarrow$ negative curvature



parabola $\times \mathbb{R}$

Q: what does image of ν look like

Q: what is resulting area

$\Rightarrow$ "curvature $= 0$."

Lemma

$$f: U \longrightarrow M \subseteq \mathbb{R}^3$$

$$\begin{array}{ccc} \operatorname{Im} D\nu\big|_u & \subseteq & T_{f(u)} M \\ \cap\! | & & \cap\! | \\ \mathbb{R}^3 & & \mathbb{R}^3 \end{array}$$

Q: can im $D\nu$ be 1-dim'l, 0-dim'l ?

(pf) $\operatorname{Im} D\nu$ spand by $\frac{\partial \nu}{\partial u_i}$ $i=1,2$ $f = f(u_1, u_2)$

Show $\langle \nu, \nu \rangle = 1$

$$2\left\langle \frac{\partial \nu}{\partial u_i}, \nu \right\rangle = 0$$

$$\Longrightarrow \frac{\partial \nu}{\partial u_i} \perp \nu$$ but ν is perpendicular to TM

$$\Longrightarrow \frac{\partial \nu}{\partial u_i} \in TM$$

$\square$

Gauss map depends on parametrization

(Bad)

$f(u) = p$

Def Shape operator, $p \in M$, $f : U \to M$ local parametrization at p

$$\begin{tikzcd} L : T_pM \arrow[rr] \arrow[dr, "Df|_u^{-1}"'] & & T_pM \\ & T_u U \arrow[r, "-D\nu|_u"'] & \operatorname{Im} D\nu|_u \arrow[u, hook] \end{tikzcd}$$

linear transformation

Lemma: L is independent of parametrization
(up to sign $\longleftrightarrow$ orientation)

(pf) Suppose $\tilde{f}$ is another parametrization

$$\tilde{f} = f \circ \varphi$$ wlog φ is orientation preserving

$$\tilde{f} \rightsquigarrow \tilde{\nu}, \tilde{L}$$

Note: $\tilde{\nu} = \nu \circ \varphi$ (orientation reversing $\tilde{\nu} = -\nu \circ \varphi$)

$$\tilde{L} = -D\tilde{\nu} \circ D\tilde{f}^{-1}$$
$$= -D(\nu \circ \varphi) \circ D(f \circ \varphi)^{-1}$$
$$= -D\nu \circ D\varphi \circ [Df \circ D\varphi]^{-1} = -D\nu \circ D\varphi \circ D\varphi^{-1} \circ Df^{-1}$$
$$= -D\nu \circ Df^{-1} = L \quad \square$$

Rmk L is determined by M + orientation

Lem L is self adjoint wrt $I(-,-)$

$$\langle LX, Y\rangle = \langle X, LY\rangle$$

(pf)

$$L\frac{\partial f}{\partial u_i} = L\, Df\, e_i = -D\nu\, Df^{-1} Df e_i$$

$$= -D\nu\, e_i$$

$$= -\frac{\partial \nu}{\partial u_i}$$

$$\left\langle L\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j}\right\rangle = \left\langle -\frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j}\right\rangle$$

$$= -\cancel{\frac{\partial}{\partial u_i}\left\langle \nu, \frac{\partial f}{\partial u_j}\right\rangle}^{\,0} + \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j}\right\rangle$$

symmetric

$$= \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j}\right\rangle$$

Defi

$$II(X,Y) := I(LX, Y)$$ second fund form

$\mathrm{III}(X,Y) := \mathrm{I}(LX, LY)$ third fund form

$\mathrm{I}, \mathrm{II}, \mathrm{III} : T_pM \times T_pM \longrightarrow \mathbb{R}$ bilinear forms

Symmetric?

$$
\begin{aligned}
\mathrm{II}(X,Y) &= \mathrm{I}(LX, Y) \\
&= \mathrm{I}(X, LY) \\
&= \mathrm{I}(LY, X) \\
&= \mathrm{II}(Y, X) \quad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{III}(X,Y) &= \mathrm{I}(LX, LY) \\
&= \mathrm{I}(LY, LX) \\
&= \mathrm{III}(Y, X) \quad \checkmark
\end{aligned}
$$

Q: is II, III non-degenerate?

Local coordinates: $f : U \to M$

$$T_u U \times T_u U \longrightarrow T_p M \times T_p M \xrightarrow[\substack{\mathrm{I} \\ \mathrm{II} \\ \mathrm{III}}]{} \mathbb{R}$$

get pairings $\mathrm{I}(-,-)$, $\mathrm{II}(-,-)$, $\mathrm{III}(-,-)$

on $T_u U = \mathbb{R}^2$

matrices

$$\mathrm{I} \sim [g_{ij}]$$

$$\mathrm{II} \sim [h_{ij}]$$

$$\mathrm{III} \sim [e_{ij}]$$

$$g_{ij} = \left\langle \frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$$

"hessian" $\quad h_{ij} = \left\langle \nu, \frac{\partial^2 f}{\partial u_i \partial u_j} \right\rangle = - \left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial f}{\partial u_j} \right\rangle$

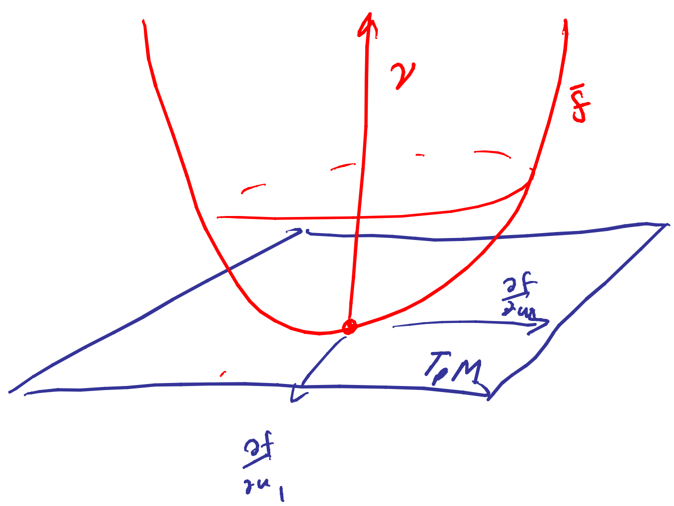
$$e_{ij} = \left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle$$

Idea: Understand L through h_{ij}

$$L\left(\frac{\partial f}{\partial u_i}\right) = \left\langle L\frac{\partial f}{\partial u_i}, \frac{\partial f}{\partial u_1}\right\rangle \frac{\partial f}{\partial u_1} + \left\langle LX, \frac{\partial f}{\partial u_2}\right\rangle \frac{\partial f}{\partial u_2}$$

So, using basis $\left\{\frac{\partial f}{\partial u_1}, \frac{\partial f}{\partial u_2}\right\}$ of T_pM

$$L \sim \left[h_{ij} \right]$$



think of M as locally modelled graph of a function

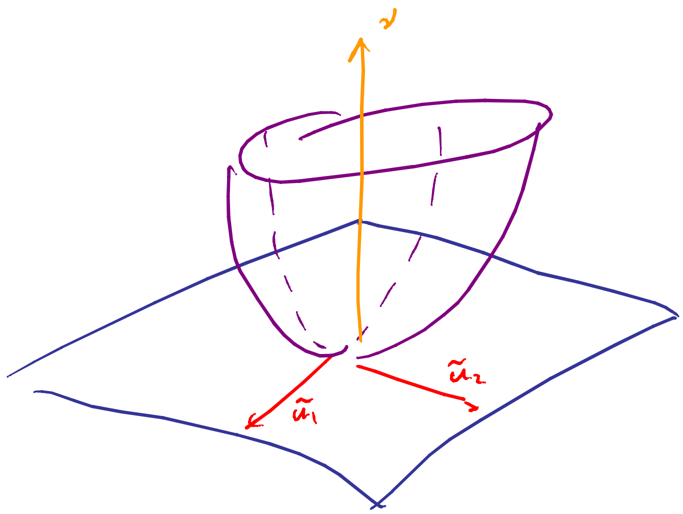
$\bar{f}(u_1, u_2)$ Then at u $\frac{\partial^2 \bar{f}}{\partial u_i \partial u_j} = h_{ij}$

h_{ij} symmetric $\Rightarrow$ diagonalizable

$\Rightarrow$ $\exists$ coordinates $\tilde{u}_1, \tilde{u}_2$

so that $\tilde{h}_{ij} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

$\lambda_1, \lambda_2 > 0$

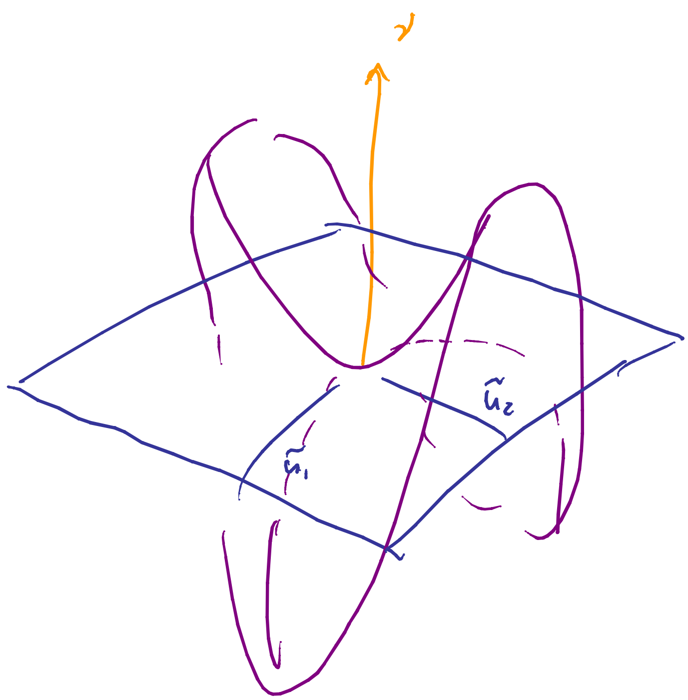


$$\frac{\partial^2 f}{\partial x_1^2} = \lambda_1$$

$$\frac{\partial^2 f}{\partial x_2^2} = \lambda_2$$

$\lambda_1 < 0$

$\lambda_2 > 0$



$\lambda_1, \lambda_2 < 0$