18.950: PSET 1

1. (4 points) Suppose that

$$f(x, y) : \mathbb{R}^2 \to \mathbb{R}$$

is C^α. Show that the surface in $\mathbb{R}^3$ given by the "graph of the function"

$$z = f(x, y)$$

is always a 2-dimensional submanifold of $\mathbb{R}^3$ of class C^α.

2. (3 points) After Definition 1.5, the book remarks that the image of an injective immersion is not always a submanifold. Give an example of an injective immersion

$$f : (a, b) \to \mathbb{R}^2$$

whose image is not a 1-dimensional submanifold of $\mathbb{R}^2$. (A well drawn picture accomponied by an intuitive explaination will suffice.)

3. (5 points) Let $M \subset \mathbb{R}^n$ be a C^∞ k-dimensional submanifold defined by

$$M = F^{-1}(0)$$

for a C^∞ map $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ of maximal rank. Show that for every $p \in M$

$$T_pM = \ker DF|_p.$$

(i.e. regarding T_pM as a subspace of $T_p\mathbb{R}^n$, show that it agrees with the kernel of the linear mapping

$$DF|_p : T_p\mathbb{R}^n \to T_0\mathbb{R}^{n-k}.$$

4. (4 points) Assume problem 3. Suppose, in the notation of problem 3, we are in one of the following cases:

a: $n = 2$, $k = 1$, or
b: $n = 3$, $k = 2$.

What does problem 3 imply about the relationship between T_pM and the gradient of F at p?

5. (5 points) Sketch (or graph using a computer) the curve C in $\mathbb{R}^2$ consisting of all points (x, y) satisfying

$$y^2 = x^3 + x^2.$$

Show that C is *not* a 1-dimensional submanifold of $\mathbb{R}^2$.

[Warning: it is *not* enough to show that the Jacobean of $F(x, y) = y^2 - x^3 - x^2$ does not have maximal rank everywhere. You need to show that there are no *other* choices of F that satisfy Definition 1.5.]

[Hint: one possible approach involves using problem 4]

Date: Assigned: 9/15/09, Due: 9/22/09.